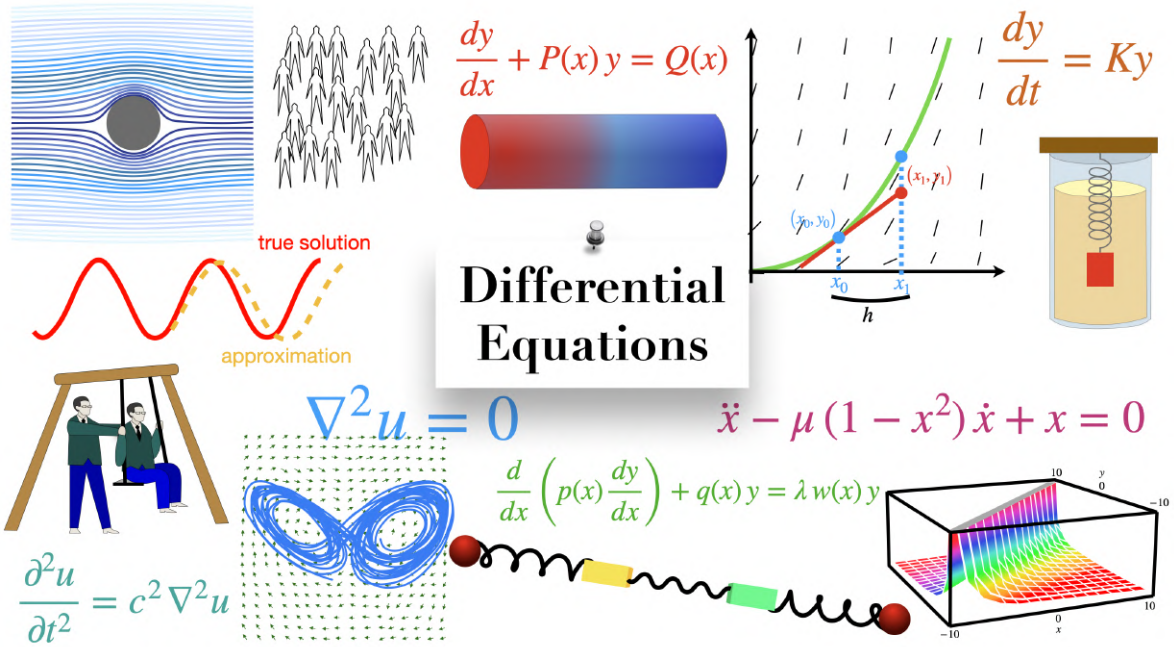
6 Steps to MASTER Differential Equations

by DiBeos



"Thus the partial differential equation entered theoretical physics as a handmaid, but has gradually become mistress" – **Albert Einstein**

If you want to have access to the FULL-PDF version, click on this link. We added a few exercises at the end (with the detailed solutions) and extra explanations throughout it. Also, I remind you guys that we provide the FULL-PDF version for free for all members of the channel. Just join us on YouTube! We'd like to keep our videos free of interruptions and sponsors, so that the sole focus is the subject at hand. But in order to do that we need your help. Thanks for supporting our work.

Introduction

What does it really take to master differential equations?

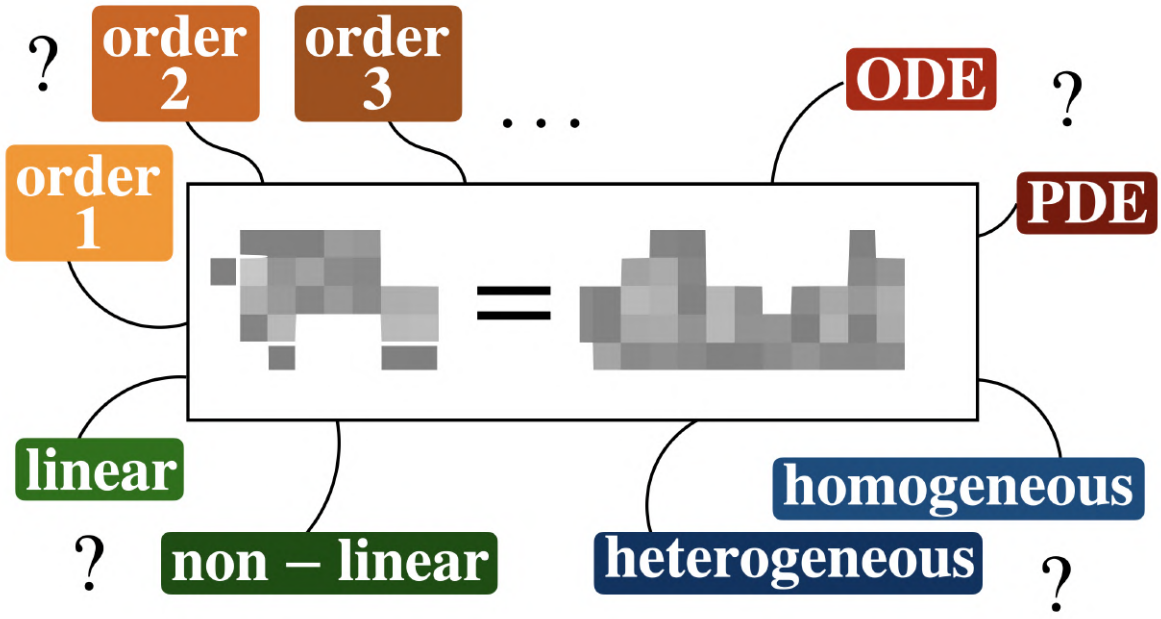
Here we will see 6 steps that will serve as a roadmap to help you master differential equations (DEs) on your own pace, and with as much depth as you like. The 6 steps that we will explore today are the following:

1 The Language of DEs
2 Exact Solutions (When Possible)
3 Rigorous Foundation of DEs
4 Qualitative Methods (When Solutions Are Intractable)
5 Numerical Methods
6 PDEs & Mathematical Physics

1 The Language of DEs

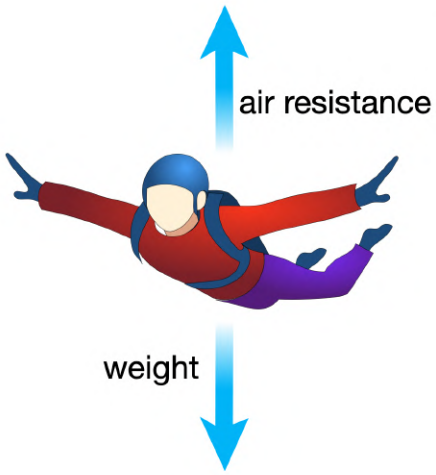
The first thing you need to master is the language of differential equations.

When you look at a DE, you should be able to immediately identify whether it's an **ordinary** or **partial differential** equation (**ODE** or **PDE**), whether it's **linear** or **nonlinear**, whether it's **homogeneous** or **heterogeneous**, and what its **order** is: first, second, third, ...



Why does it matter? Because your ability to choose the right method depends on how you classify the equation. But that's not enough. You also need to know what kind of problem you're trying to solve. Is it an **initial value problem** (**IVP**)?

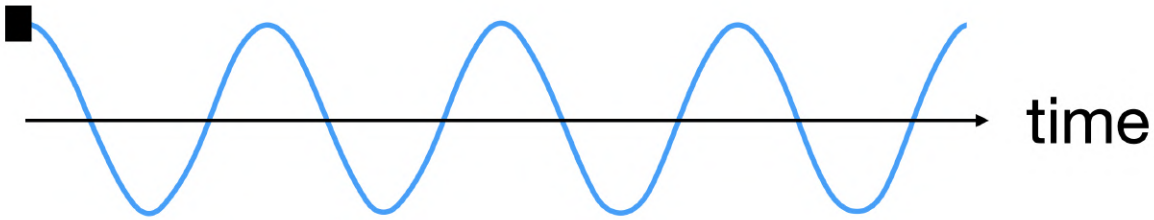
initial value problem (IVP)?



$$\frac{dv}{dt} = -g - \frac{k}{m}v$$

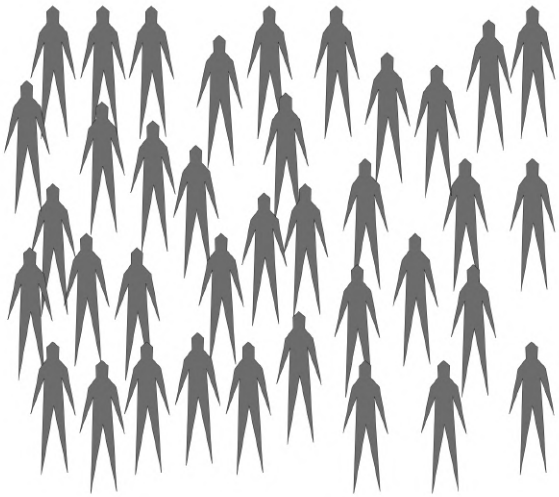
$$v(0) = 0 \quad \text{(IVP)}$$

Like a free fall with a fixed initial velocity?



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \begin{matrix} u(x,0) = f(x) \\ \frac{\partial u}{\partial t}(x,0) = g(x) \end{matrix} \quad \text{(IVP)}$$

Or a wave oscillating with initial and final time described only by functions of space?

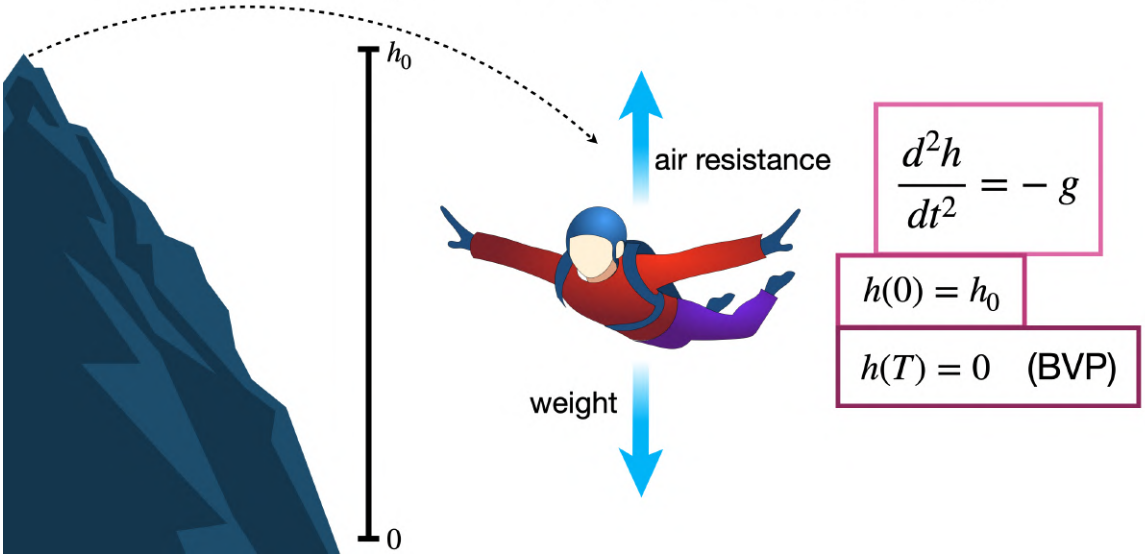


$$\frac{dy}{dt} = r\,y\left(1 - \frac{y}{K}\right) \qquad y(0) = y_0 \quad \text{(IVP)}$$

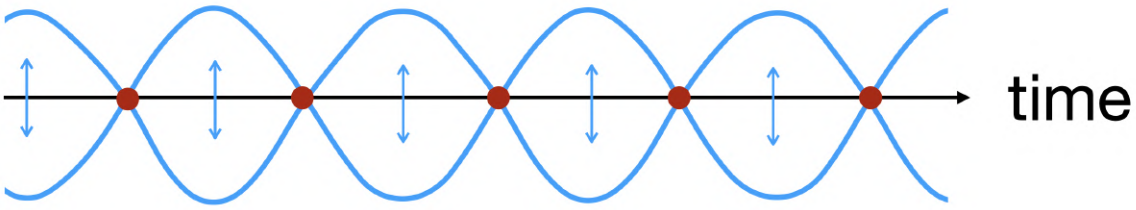
Or maybe it is a population growth model with a fixed initial number of members?

Is it a **boundary value problem** (**BVP**)?

boundary value problem (BVP)?

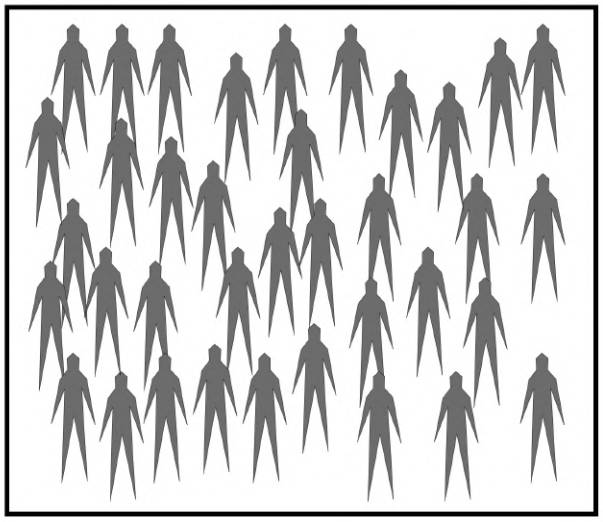


Like a free fall with fixed initial altitude and final destination?



$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad u(0,t) = 0 \qquad u(L,t) = 0 \qquad \text{(BVP)}$$

Or a stationary wave with fixed boundaries?



$$\frac{d^2y}{dx^2} + r\,y\left(1 - \frac{y}{K}\right) = 0$$

$$y(0) = y(L) = 0 \quad \text{(BVP)}$$

Or a population growth that has a physical limitation in space? Like walls surrounding it.

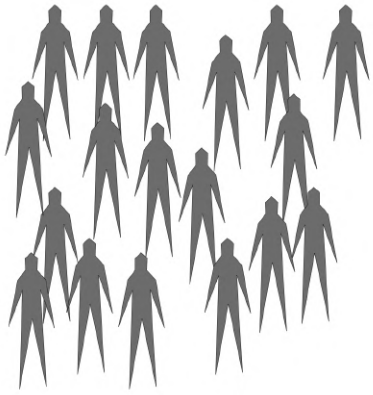
Or maybe it's possibly both? Initial AND boundary value problem. Or maybe neither! Understanding first the type of equation, and the type of problem, gives you the first part of the roadmap to look for solutions.

1. Language of Differential Equations Types of DE's : ODEs vs PDEs, linear vs non-linear, homogeneous vs heterogeneous, order (1st, 2nd 3rd...) Types of problems : IVP's, BVP's,... Goal : classification and terminology	2.	3.
4.	5.	6.

So that's your goal in the first stage: Get fluent in DE's *classification* & *terminology*.

2 Exact Solutions (When Possible)

Once you know how to classify a DE using the appropriate terminology, the next step is to **solve it**, when possible. For *first-order* equations, you should learn classic techniques like **separation of variables**,



1st order

$$F\left(x, y, y'\right) = 0$$

separation of variables

$$\frac{dy}{dt} = Ky \implies \frac{dy}{y} = Kdt$$

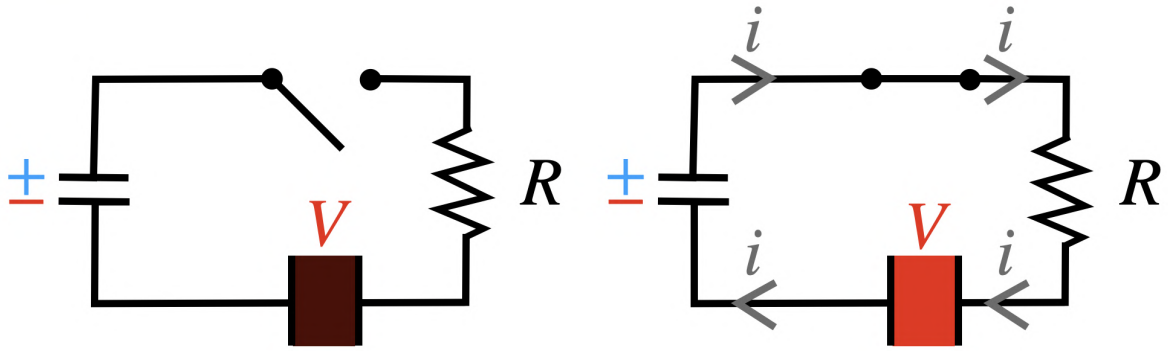
like in this simple example, where we separate y in the left-hand side from t in the right-hand side.

You should also know how to **integrate factors**,

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

integrating factor $\Downarrow$ $\mu(x) = e^{\int P(x)\,dx}$

$$\mu(x)\frac{dy}{dx} + \mu(x)\,P(x)\,y = \frac{d}{dx}\left[\mu(x)\,y\right]$$



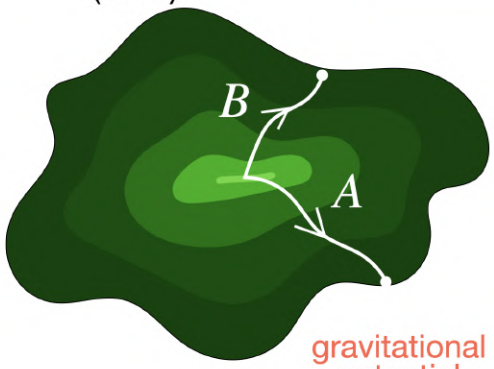
like in this resistor-capacitor circuit that is discharging and charging in a periodic fashion.

And you should also know methods for **exact** and **homogeneous equations**,

exact and homogeneous equations

$$W_A = W_B$$

(work)



$$M(x,y)\,dx + N(x,y)\,dy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\Downarrow$$

gravitational potential energy

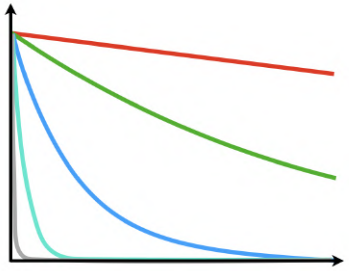
$$d\Psi = M\,dx + N\,dy$$

like when studying problems involving gravitational potential energy, for example.

This is like a pattern recognition game: once you spot the structure, you know which method to apply. But you must train your eyes first. These techniques also apply to problems involving *exponential decay*, or *Newton's law of cooling*, for example.

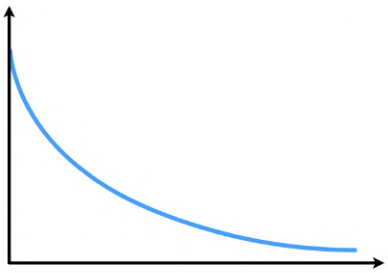
Exponential decay

$$\frac{dN}{dt} = -\lambda N$$



Newton's law of cooling

$$\frac{dT}{dt} = -k\left(T - T_e\right)$$

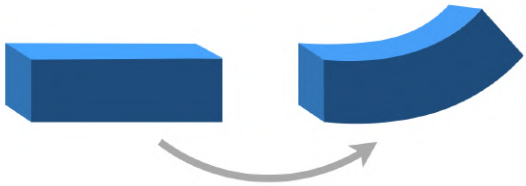


And even though these are examples in physics, you will see throughout this presentation how useful it is to learn DEs with at least a minimal knowledge of how to apply these concepts to real-life scenarios. This subject becomes really complex, really fast, so keeping a certain level of intuition based on natural phenomena might be helpful, even if you are a pure mathematician.

When we move to *second-order* equations (and beyond), things get a bit more technical. These equations are especially useful in physics, where many processes can be modeled using second-order DEs, but rarely with higher orders, even though there are a few cases.

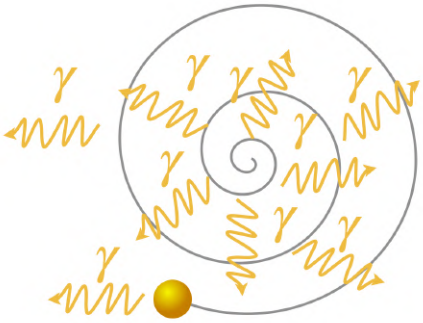
Euler – Bernoulli beam theory

$$\frac{d^4y}{dx^4} = \frac{q(x)}{EI}$$



Abraham – Lorentz force

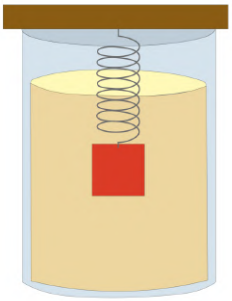
$$m\ddot{x} = F_{\text{ext}} + \frac{q^2}{6\pi\varepsilon_0 c^3}\dddot{x}$$

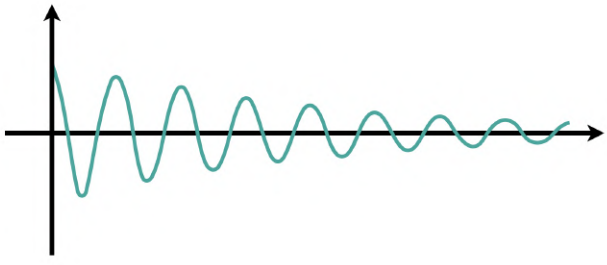


Here, you'll need methods such as the **characteristic equation** (for **linear constant-coefficient ODEs**),

characteristic equation

$$\boxed{y''} + 5\boxed{y'} + 6\boxed{y} = 0 \implies \boxed{r^2} + 5\boxed{r} + 6\cdot\boxed{1} = 0$$

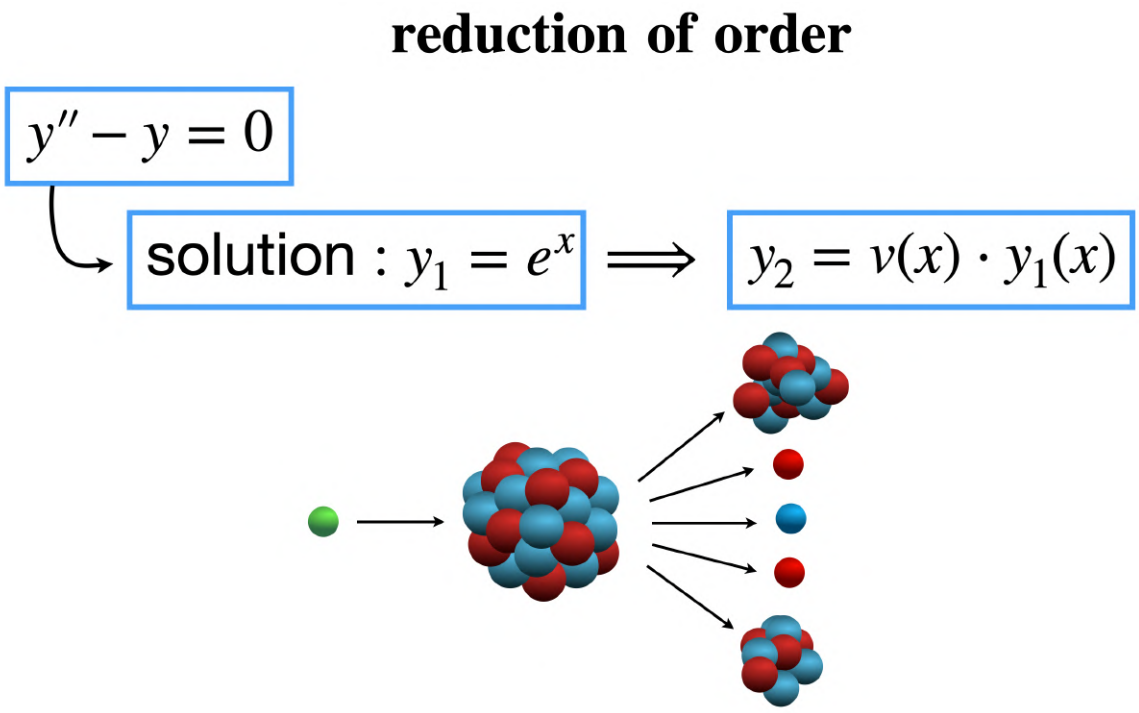




like for a damped spring.

*If you want to see the solution of the equation above, using the **characteristic equation**, check out the FULL-PDF version. We also added a few exercises at the end (with the detailed solutions). Thanks for supporting our work.*

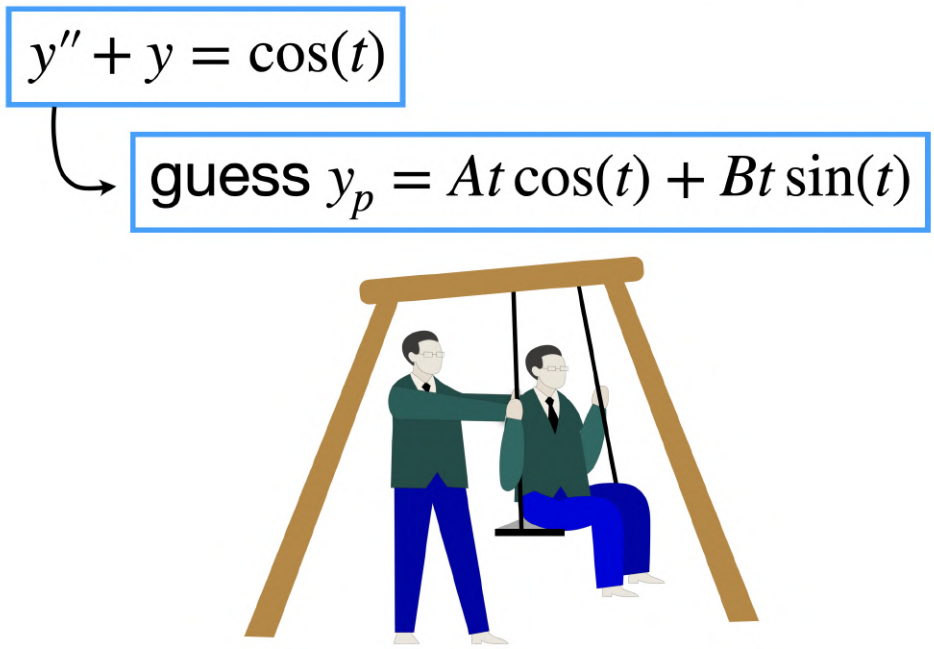
Another method is **reduction of order**,



like in a chain reaction in nuclear fission.

*We also added the solution of the equation above, using **reduction of order** in the FULL-PDF version.*

You might also need to know how to work with **undetermined coefficients**,



like in a forced undamped harmonic oscillator.

Once again, check out the FULL-PDF to see the detailed solution of the equation above, using ***undetermined coefficients****.*

Or even the method of **variation of parameters** to handle nonhomogeneous terms,

variation of parameters

$y'' + y = \tan(t)$ ⟶ homogeneous:

$y_h = A\cos(t) + B\sin(t)$

⇓

particular:

$y_p = u_1(t)\cos(t) + u_2(t)\sin(t)$

like in a forced undamped harmonic oscillator, with irregular forcing.

*Solution above using **variation of parameters**: FULL-PDF.*

Therefore, that's your goal in the second stage: be able to *recognize solvable equations*, and then solve them.

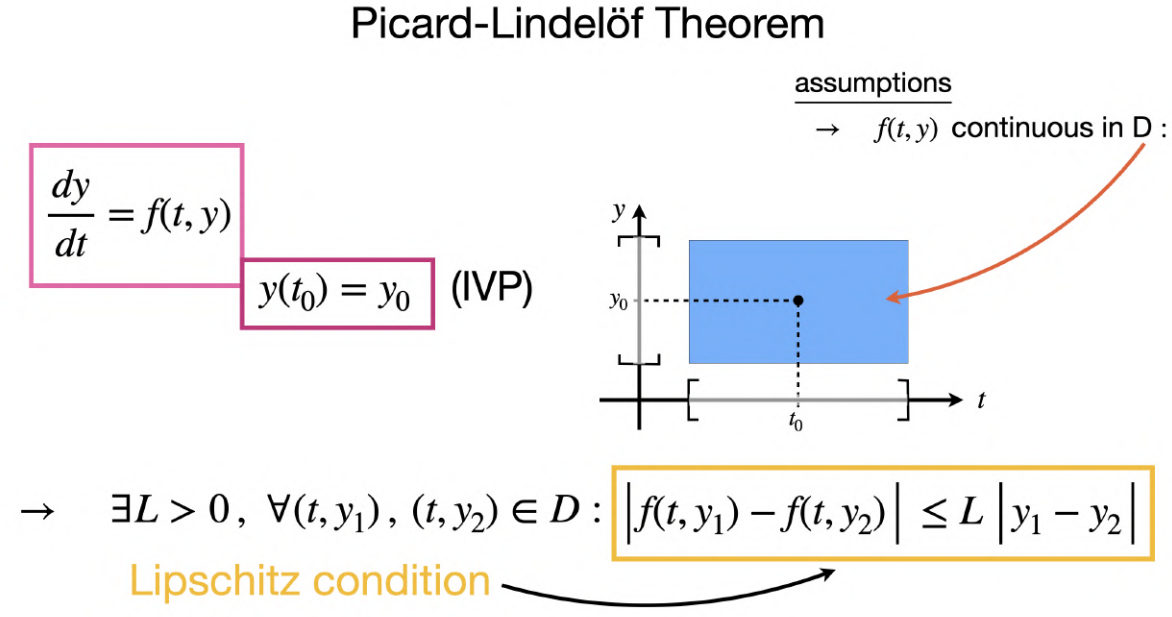
1. Language of Differential Equations Types of DE's : ODEs vs PDEs, linear vs non-linear, homogeneous vs heterogeneous, order (1st, 2nd 3rd...) Types of problems : IVP's, BVP's,... Goal : classification and terminology	**2. Exact Solutions (when possible)** 1^{st} order : separation of variables, integrating factors, homogeneous and exact equations. 2^{nd} order : characteristic equations, reduction of order, undetermined coefficients, variation of parameters. Goal : recognize solvable equations	3.
4.	5.	6.

3 Rigorous Foundation of DEs (Existence, Uniqueness, Behavior)

At this point, I have a confession to make. This right here should be the second point, because before looking for solutions of a specific DE, you should first make sure that solutions even exist, whether they are unique or not, and what kind of behavior they might have. Of course, I wanted to present the simple case of finding exact solutions first, because I believe that the theory of DEs becomes much easier to understand this way.

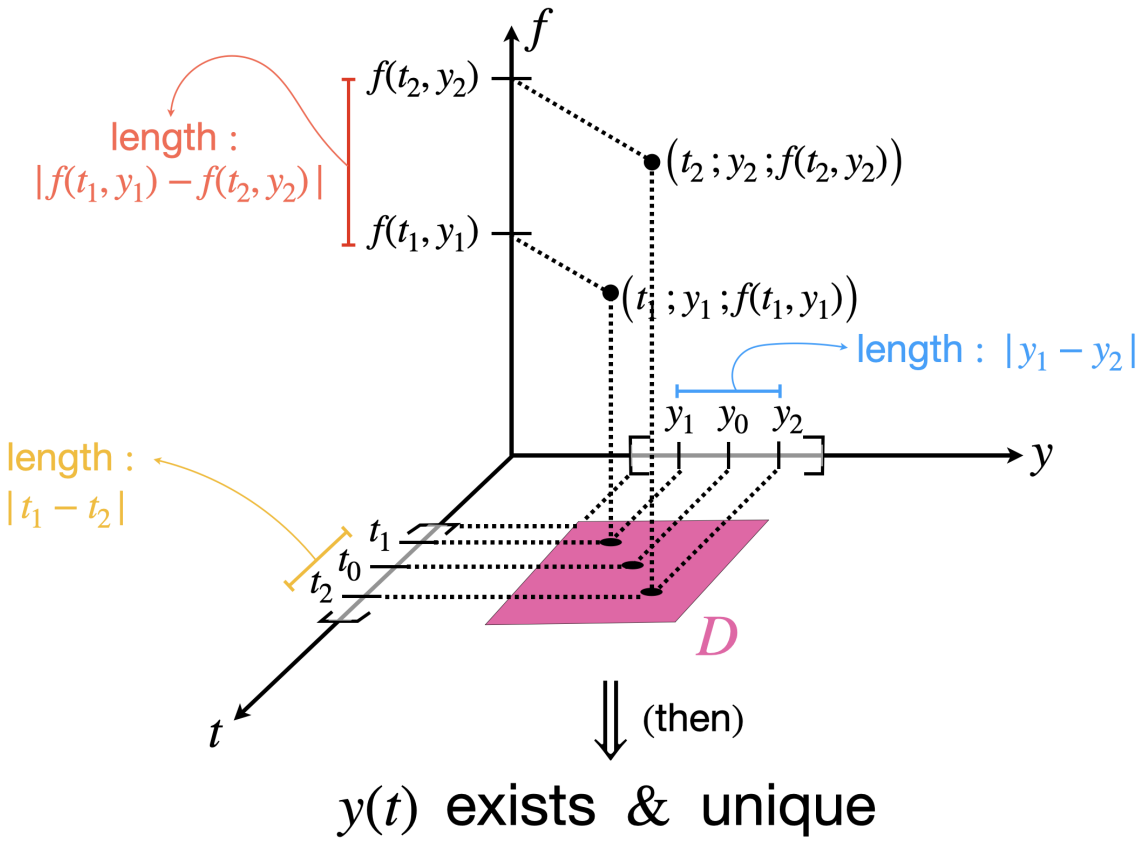
I remind you guys that our favorite method (i.e. the one we, the DIBEOS, always highly recommend you implement when learning any math field) is to first get **(1) Intuition**, then look at and play with **(2) Concrete Examples**, then study the **(3) Rigor of the subject**, and finally **(4) Practice** by solving a bunch of exercises. So it makes sense to first learn the language of DEs, and solve some examples before getting into the rigorous theory itself.

Now we're ready to explore the rigorous theoretical foundation of DEs.



The **Picard-Lindelöf theorem** guarantees *existence* and *uniqueness* under certain smoothness conditions. In this initial value problem (be-

low), we assume the function to be continuous in a certain rectangular domain, and we assume what is called the **Lipschitz condition**.



Then ($\Longrightarrow$) the theorem says that there exists a unique solution. We won't get into the details here, but the Lipschitz condition is exactly what prevents the function from changing too drastically in the vertical direction.

Another important result is **Peano's theorem**, which assures *existence* of solutions even when uniqueness can't be guaranteed.

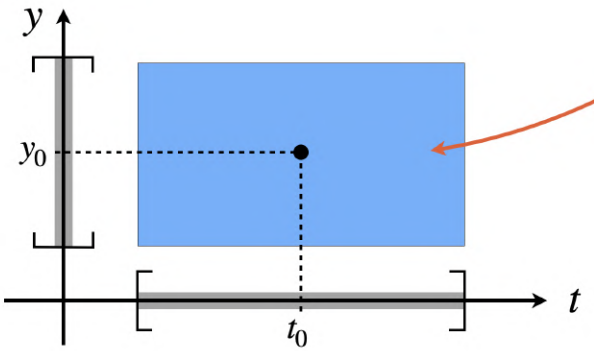
Peano theorem

$$\frac{dy}{dt} = f(t, y)$$

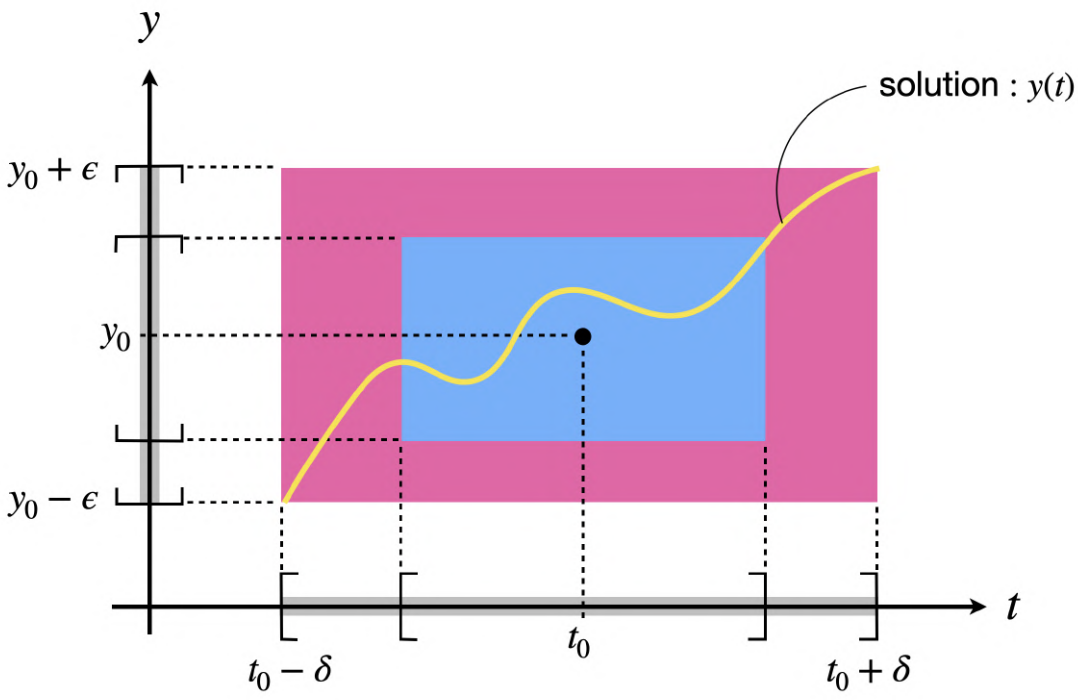
$$y(t_0) = y_0 \quad \text{(IVP)}$$

assumptions

→ $f(t, y)$ continuous in D :



It assumes that f is continuous in a rectangular domain, just as before. But there is no Lipschitz condition this time.



And the implication is that there is at least one solution, locally.

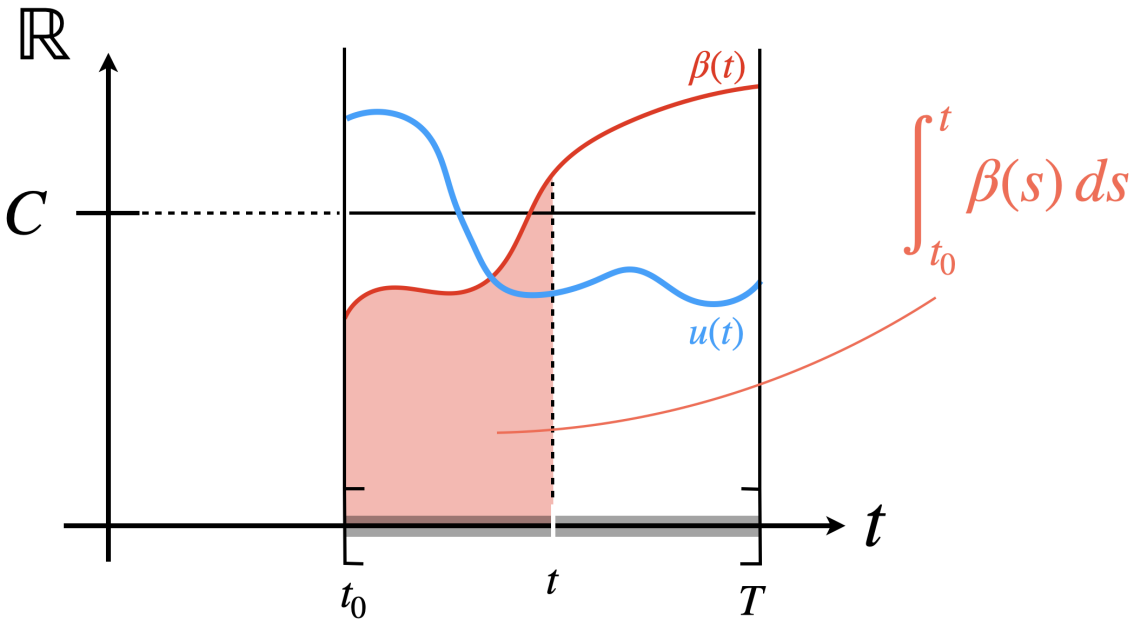
$\Downarrow$ (then)

$\exists\, y(t)$ locally, i.e.,

$[t_0 - \delta, t_0 + \delta] \quad (\delta > 0 \text{ small})$

Beyond existence, we need to control and understand the growth (or shape) of solutions. For example, **Grönwalls's inequality** helps us to establish bounds on how solutions evolve, especially in stability theory.

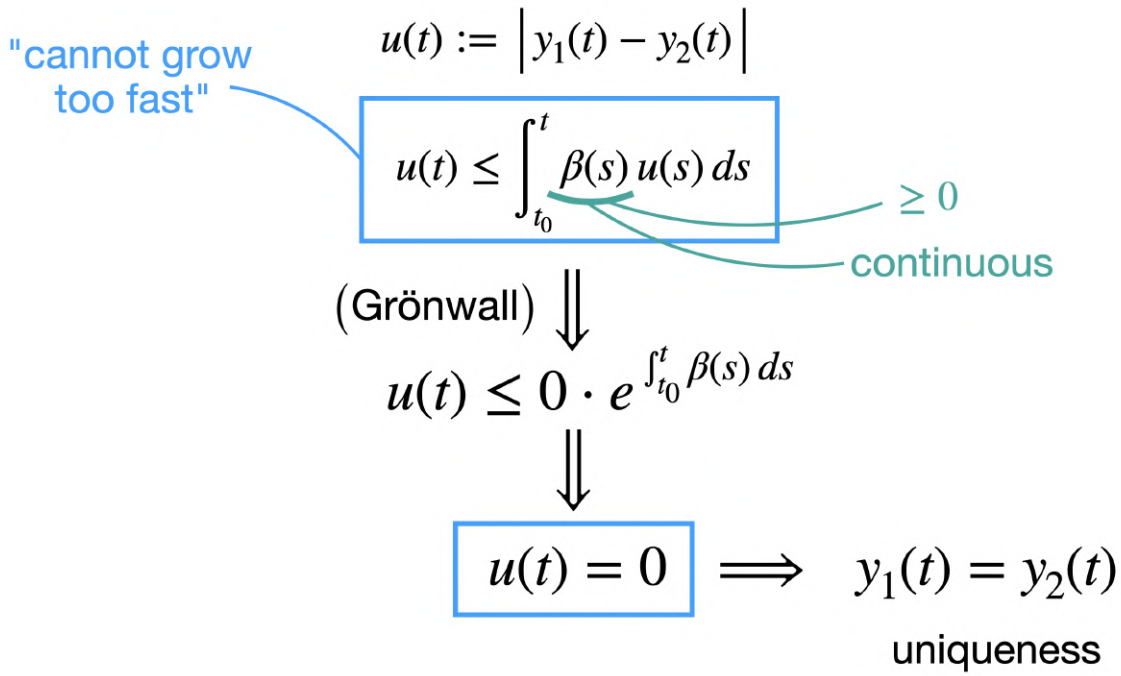
Grönwalls's inequality



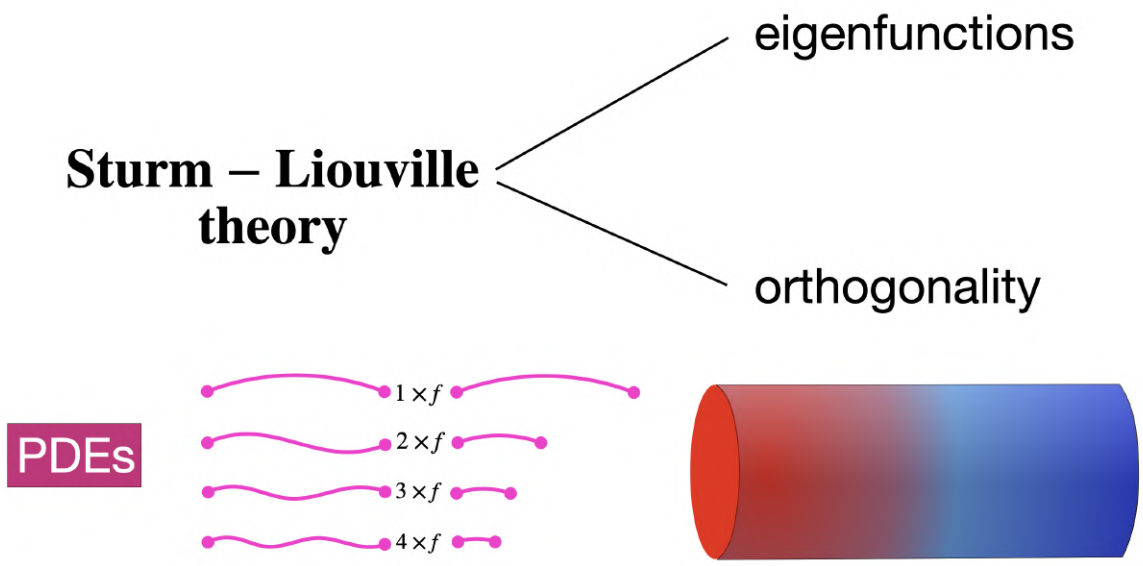
$$u(t) \leq C\, e^{\int_{t_0}^{t} \beta(s)\, ds}$$ (special case)

This is typically used when you have a bound on a function (say, the difference between two solutions) and want to show that it cannot

grow too fast, or even that it must vanish, which implies that the solution is unique.



After that, we have: **Sturm-Liouville theory**, which deals with the actual structure of solutions. In particular, *eigenfunctions* and *orthogonality*, which are very useful when trying to solve PDEs, like in systems involving vibrating strings and heat diffusion.



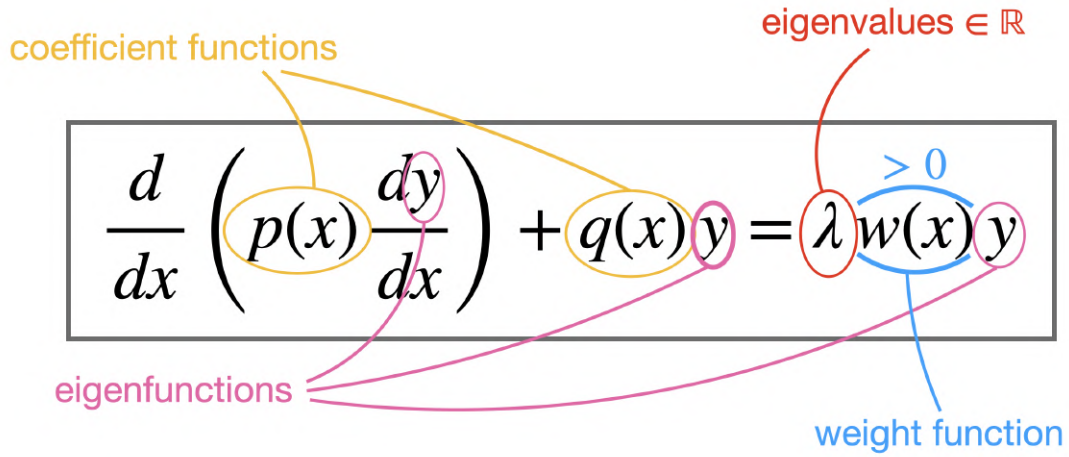
A Sturm-Liouville problem is a second-order linear differential equation of this form:

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y = \lambda\, w(x)\, y$$

With the following boundary conditions:

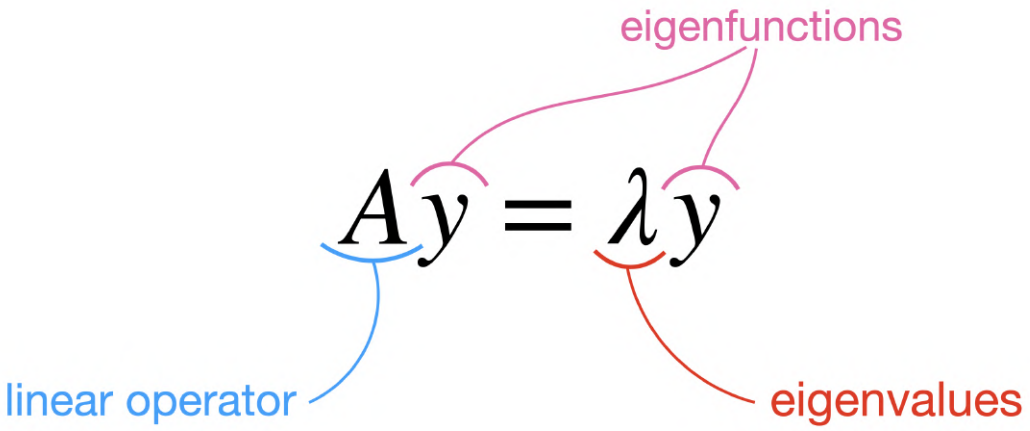
Sturm – Liouville problem

$$\text{boundary conditions} \begin{cases} \alpha_1\, y(a) + \alpha_2\, y'(a) = 0 \\ \beta_1\, y(b) + \beta_2\, y'(b) = 0 \end{cases}$$

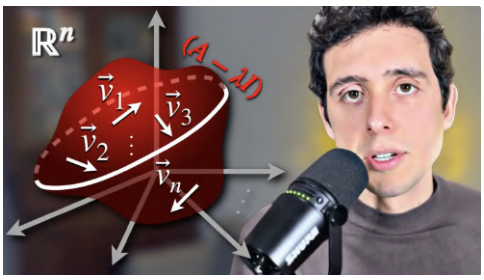


$p(x)$ and $q(x)$ are called coefficient functions. y are the eigenfunctions. λ are the eigenvalues. And $w(x)$ is called the weight function.

I know, looking at this equation (written this way), it's hard to tell that this is a *linear eigenvalue problem*. Maybe you're more used to the linear operator language:



By the way, If you want to reinforce your intuition behind eigenvalue problems in Linear Algebra, check out this video and PDF:



The Core of Eigenvalues & Eigenvectors

We can rewrite the equation this way (below), and then we clearly see that this is the operator. The operator A, as we defined here, is not only linear, but also a **self-adjoint operator**.

$$\boxed{\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)\,y = \lambda\, w(x)\,y}$$

$$\Downarrow$$

$$\frac{1}{w(x)}\left(\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + q(x)\,y\right) = \lambda\, y$$

$$\Downarrow$$

$$\boxed{\frac{1}{w(x)}\left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right] + q(x)\right)}\, y = \lambda\, y$$

$$\Vdots$$

A (linear operator)
self-adjoint

A self-adjoint operator is a linear operator with real eigenvalues λ. Each of these eigenvalues $\lambda_1 < \lambda_2 < \lambda_3 < \ldots \to \infty$ has their own eigenfunctions $y_1, y_2, y_3, \ldots \to \infty$ associated with each one of them.

$$\boxed{\frac{1}{w(x)}\left(\frac{d}{dx}\left[p(x)\frac{d}{dx}\right]+q(x)\right)}\, y = \lambda\, y$$

$$\vdots\!\!\| \quad A \text{ (linear operator)}$$
self-adjoint

eigenvalues $\quad \lambda_1 < \lambda_2 < \lambda_3 < \cdots \to \infty$

eigenfunctions $\quad y_1,\ y_2,\ y_3,\ \cdots \to \infty$

If you want to learn very useful intuitions behind the concepts of ***self-adjoint operators***, ***Hilbert spaces*** *and* ***orthogonality*** *in this context, check out the FULL-PDF version.*

So, here's the real power of the Sturm-Liouville theory: once you've found the eigenfunctions $\{y_n(x)\}$ and eigenvalues $\{\lambda_n\}$ of the self-adjoint operator A, you've essentially "diagonalized" the operator.

eigenfunctions $\{y_n(x)\}$ eigenvalues $\{\lambda_n\}$

"diagonalized"

$$A = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

And since these eigenfunctions form an orthogonal basis for the Hilbert space of solutions, we can express any solution $u(x)$ of a wide range of DEs (including PDEs!) as an *infinite linear combination of the eigenfunctions*:

infinite linear combination

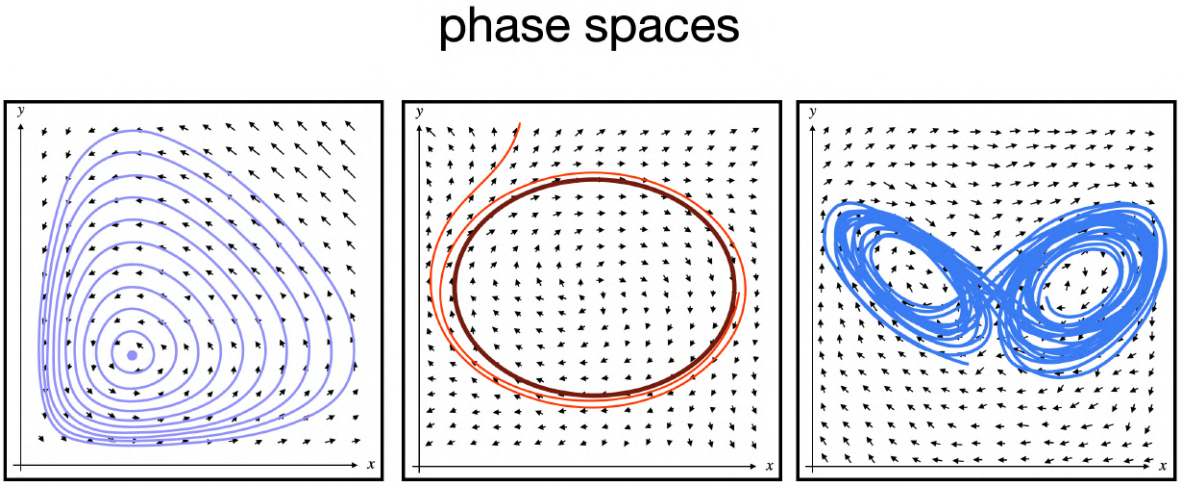
$$u(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots$$

initial or boundary conditions

Where each coefficient C_n $(n \in \mathbb{N})$ depends on the initial or boundary conditions.

Finally, for *nonlinear* systems, **phase spaces** are particularly important, because they allow us to visualize complex behavior (like chaos, for

example). Instead of solving explicitly, we study the geometric behavior of solutions, such as *equilibria*, *limit cycles*, *attractors*, and so on.



These tools tell us what's possible even before we try to compute anything.

Of course, we couldn't cover all the important results here, but if you master just the ones we talked about you'll be way ahead of most people when it comes to knowing how to tackle problems involving DEs.

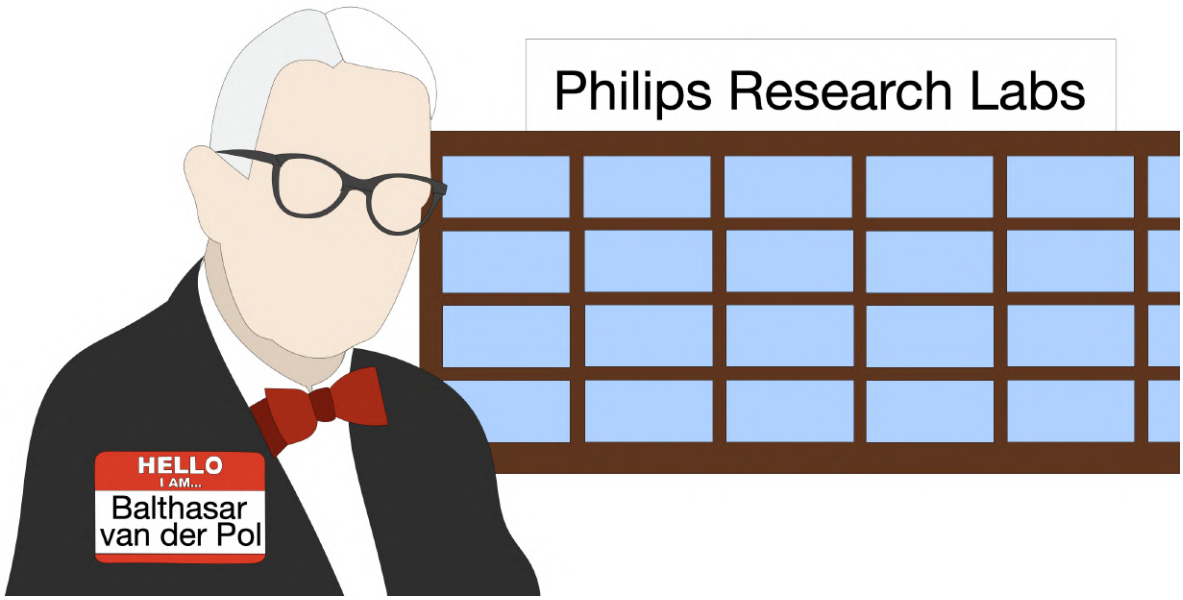
1. **Language of Differential Equations** Types of DE's : ODEs vs PDEs, linear vs non-linear, homogeneous vs heterogeneous, order (1st, 2nd 3rd...) Types of problems : IVP's, BVP's,... Goal : classification and terminology	2. **Exact Solutions (when possible)** 1^{st} order : separation of variables, integrating factors, homogeneous, and exact equations 2^{nd} order : characteristic equations, reduction of order, undetermined coefficients, variation of parameters Goal : recognize solvable equations	3. **Rigorous Foundation of DEs (Existence, Uniqueness and Behavior)** Key results : Picard-Lindelöf theorem (existence and uniqueness), Peano's theorem, (existence only), Grönwall's inequality, (control growth of solutions), Sturm-Liouville theory (eigenvalues and eigenfunctions), phase spaces (especially for nonlinear systems) Goal : understand when solutions exist, how many, and what they look like
4.	5.	6.

4 Qualitative Methods (When Solutions Are Intractable)

Sometimes, solving a DE analytically is impossible, even though solutions do exist. In those cases, the best we can do is study the features of solutions, in order to draw insights and predict behavior.

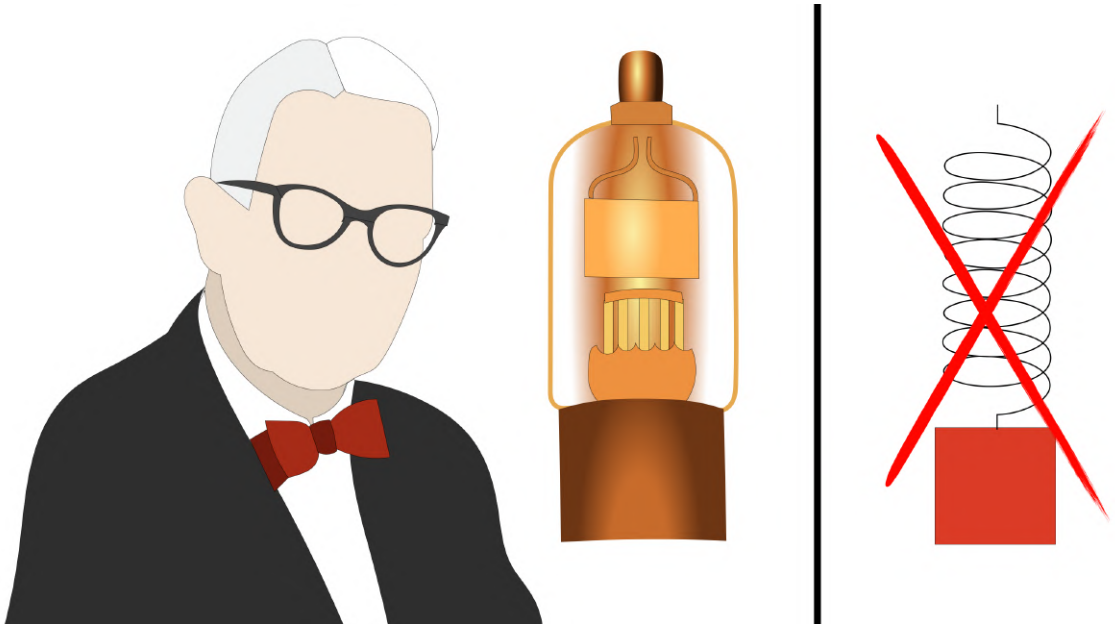
A great way of doing that is through **direction fields** and **phase portraits**. Now, we already talked briefly about phase spaces, and at this moment we're introducing direction fields and phase portraits, but what's the difference between them? Many people use these names interchangeably, as if they were the same. But if you're serious about mastering DEs, you must understand their distinction. Let's see an illustration:

The **Van der Pol Oscillator**. You can tell by this guy's name that he was Dutch.

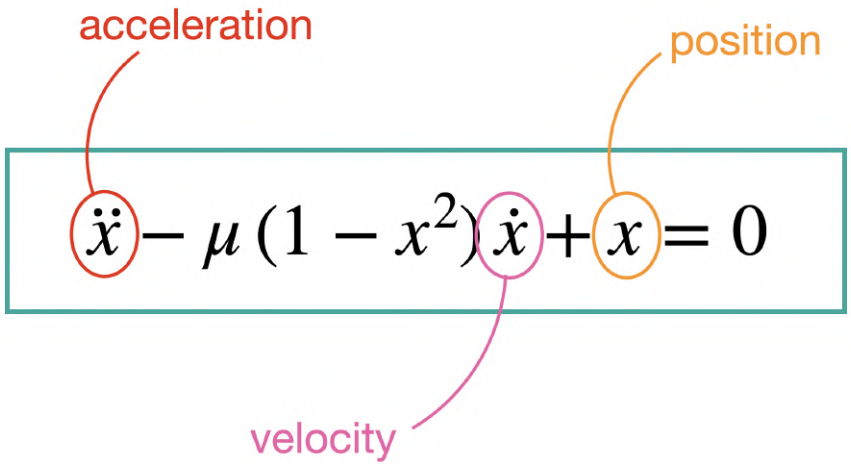


Balthasar van der Pol worked at *Philips Research Labs* in the early 20th century as a physicist and electrical engineer. While studying vacuum tubes (the kind used in old radios and amplifiers), he noticed a weird type of self-sustained oscillation, like a feedback loop, but nonlinear, so it couldn't be modeled by the standard harmonic oscillator. This

oscillation didn't die out or explode in amplitude. Instead, it settled into a steady rhythm: something he later called a *relaxation oscillation*.



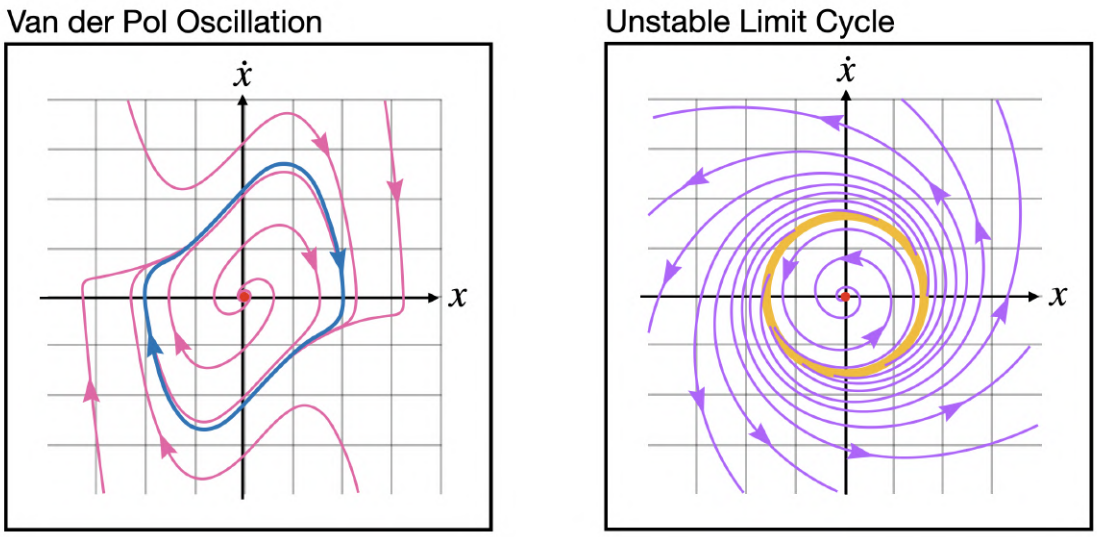
To model this phenomenon mathematically, he derived the now-famous Van der Pol equation:



This is a linear second-order differential equation that's simple to write, but impossible to solve explicitly. It's often rewritten as a first-order system:

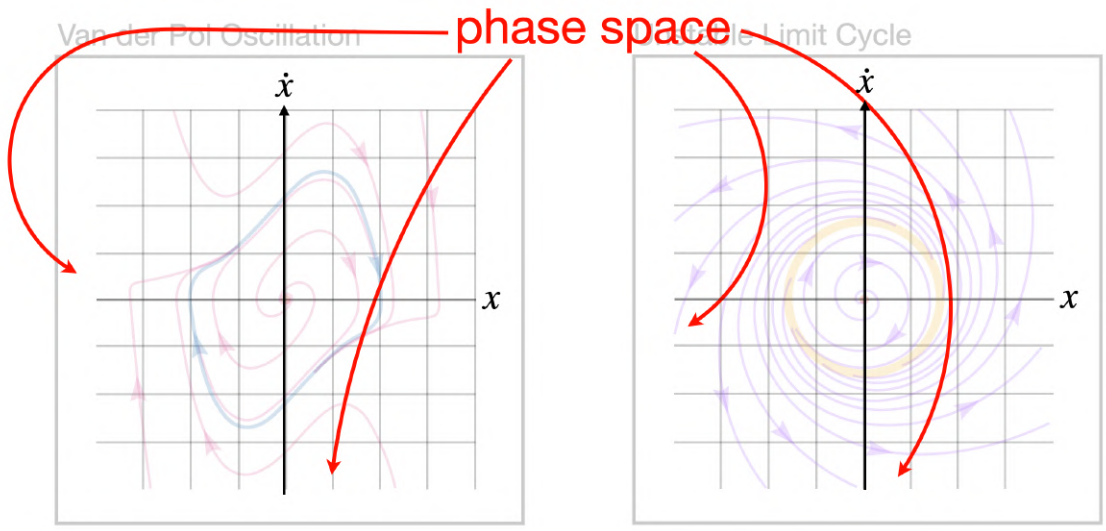
$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu\,(1 - x^2)\,y - x \end{cases}$$

This equation introduces the concept of *limit cycles*, which are closed trajectories, in a phase space, that attract or repel nearby motion.



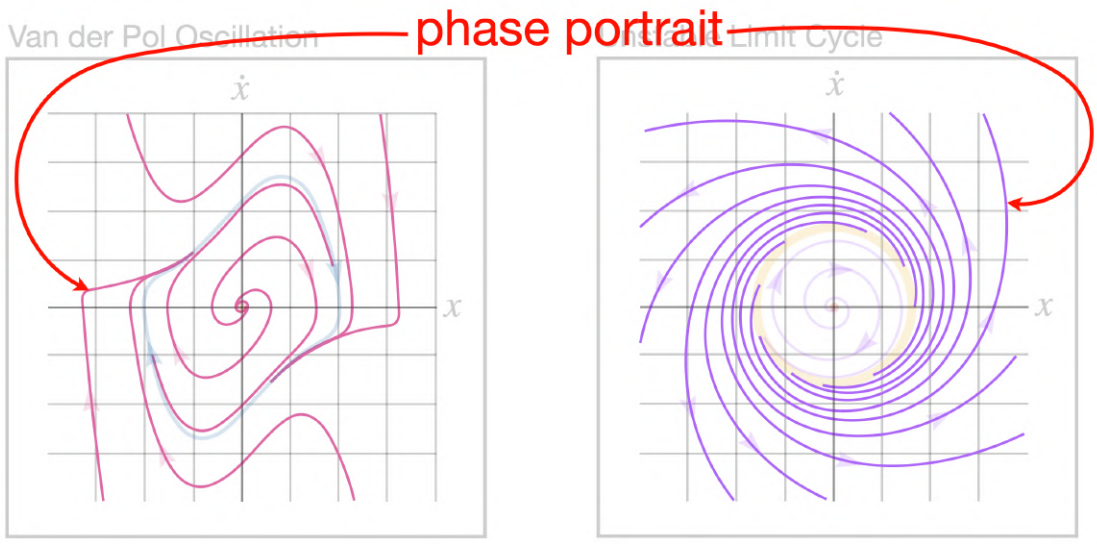
* For Van der Pol systems stable limit cycles are never repulsive.

The *phase space* is the background stage where everything happens: the $2D$ plane $\mathbb{R}^2$, with the horizontal axis representing position x, and the vertical one representing velocity $\dot{x}$.



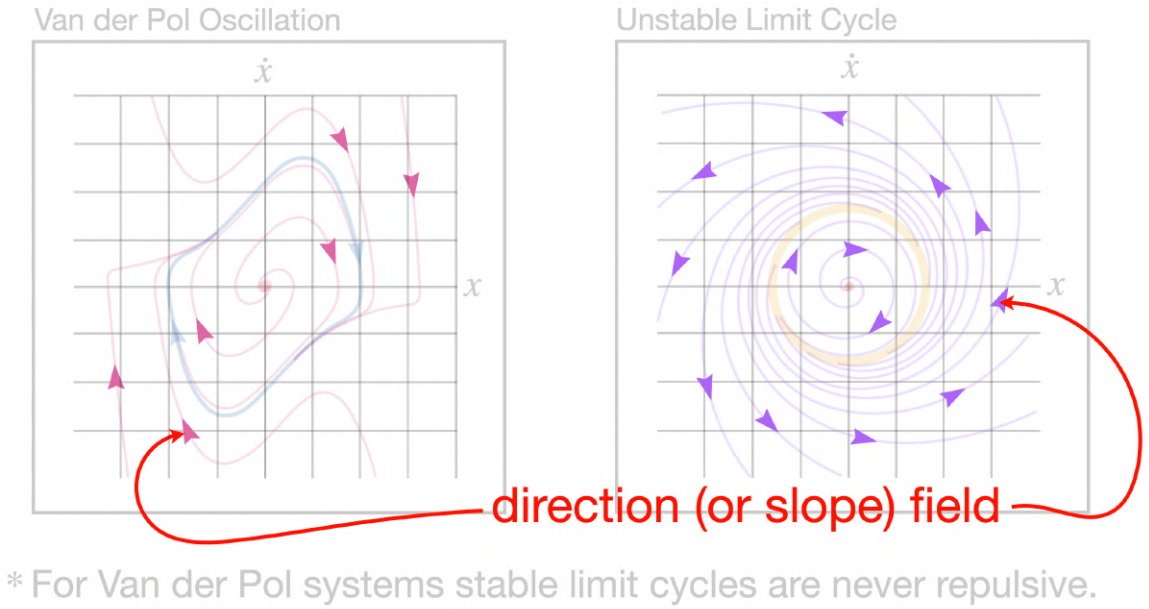
* For Van der Pol systems stable limit cycles are never repulsive.

These curves here (below) compose what we call the *phase portrait* of the system. These are solution trajectories.

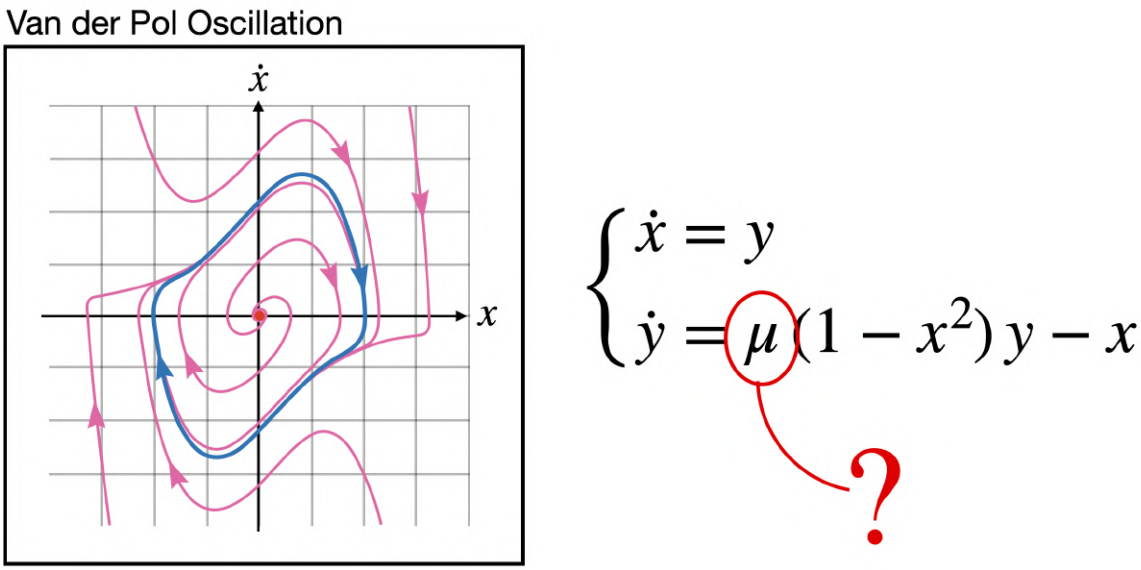


* For Van der Pol systems stable limit cycles are never repulsive.

The arrows (or vectors) make up what we call the *direction (or slope) field*, which indicates the slope of each solution curve at every point: essentially, the vector field associated with the system.

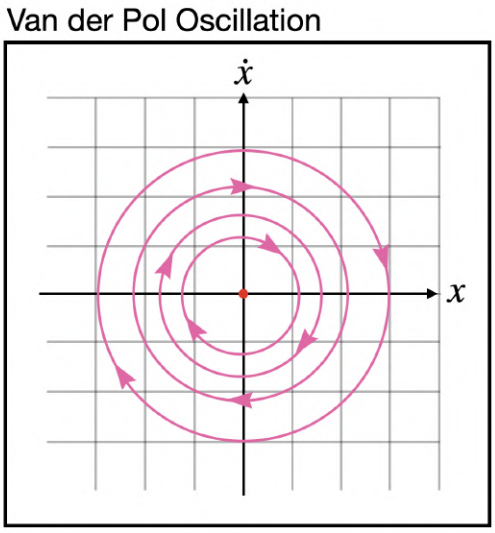


But still! Knowing all of this is useless, if just by looking at this diagram you're not able to interpret the equation accordingly. What does the parameter μ do?



μ controls both the *nonlinearity* and the *damping* (so the fact that the amplitude of oscillations decreases over time).

If $\mu = 0$: the system becomes the classic simple harmonic oscillator, with purely circular motion in the phase space.

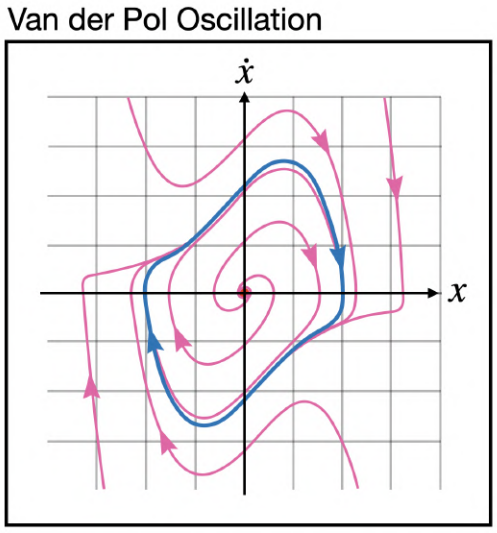


$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases}$$

$\mu = 0$: simple harmonic oscillator

* Just visual representations.

If $\mu > 0$: the system has nonlinear damping. Small oscillations grow, and large ones shrink, until the system stabilizes on a limit cycle, which is a closed loop that attracts all nearby trajectories.



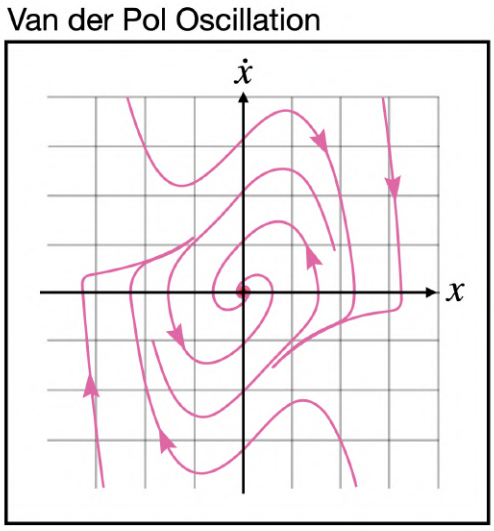
$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu(1 - x^2)y - x \end{cases}$$

$\mu = 0$: simple harmonic oscillator

$\mu > 0$: nonlinear damping

* Just visual representations.

If $\mu < 0$: The system has positive damping for all amplitudes. The origin $(x, \dot{x}) = (0, 0)$ is a globally stable fixed point, specifically a stable spiral or node. Trajectories spiral inward, toward the origin.



$$\begin{cases} \dot{x} = y \\ \dot{y} = \mu\,(1 - x^2)\,y - x \end{cases}$$

$\mu = 0$: simple harmonic oscillator

$\mu > 0$: nonlinear damping

$\mu < 0$: stable fixed point

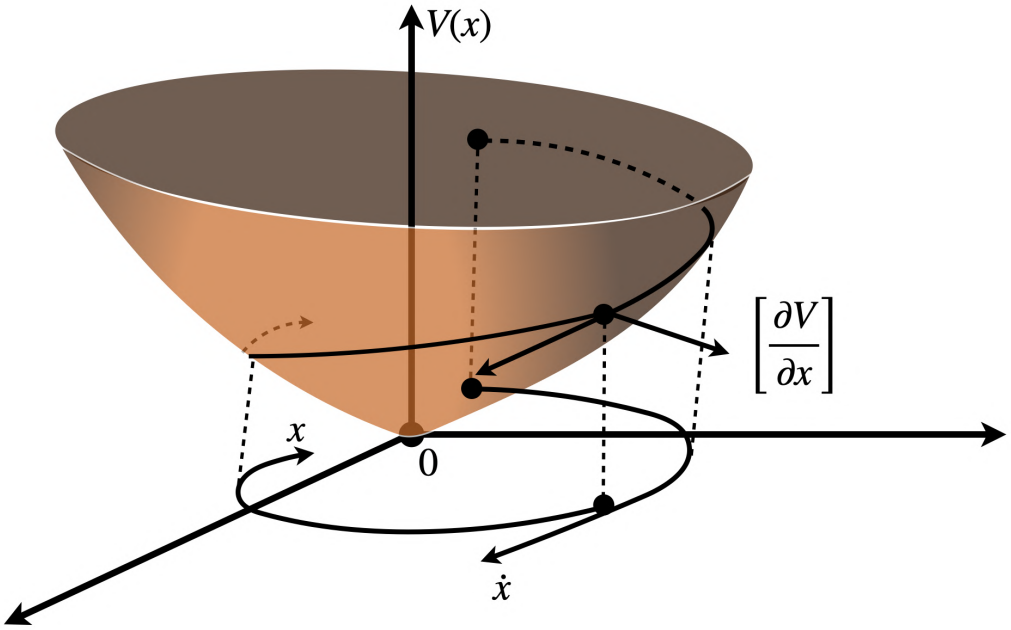
* Just visual representations.

Another important insight you can get from qualitative analysis about **stability**, that is, how sensitive a system is to small changes in its initial state.

A common method is approximating nonlinear systems near equilibrium points by a linear version of it (**linearization**).

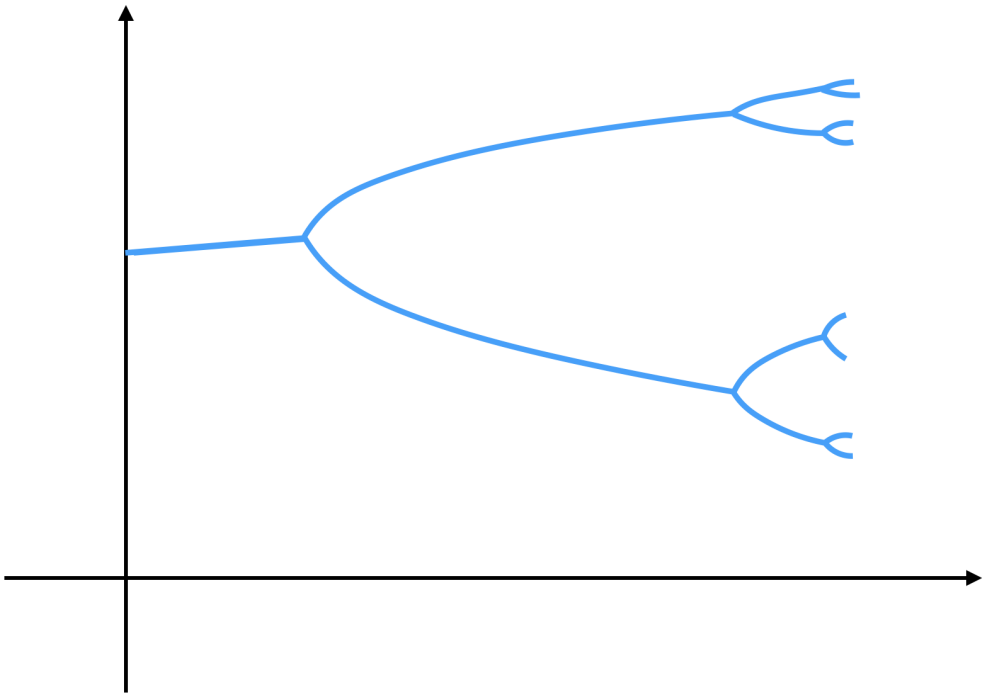
Another method is the usage of **Lyapunov functions**, which are scalar functions that help us assess the stability of equilibrium in dynamical systems without solving it explicitly.

Lyapunov functions

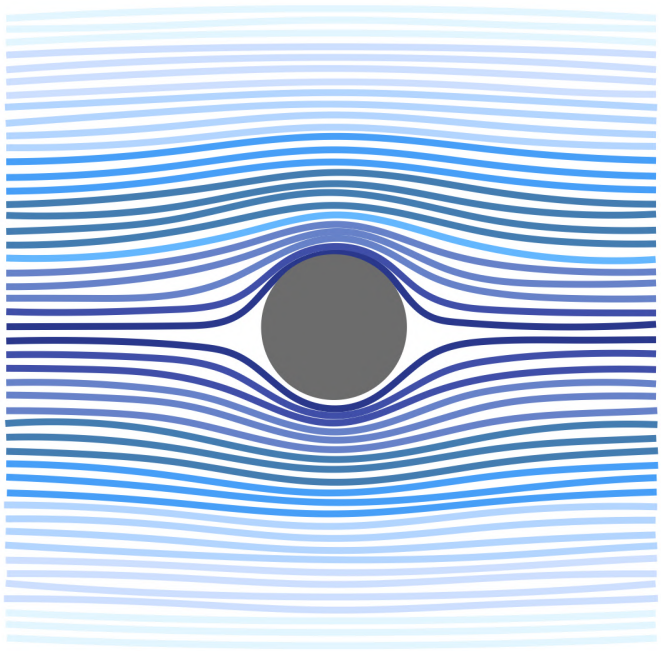


The last qualitative method for studying DEs, that we'll talk about, is **bifurcation theory**.

bifurcation theory

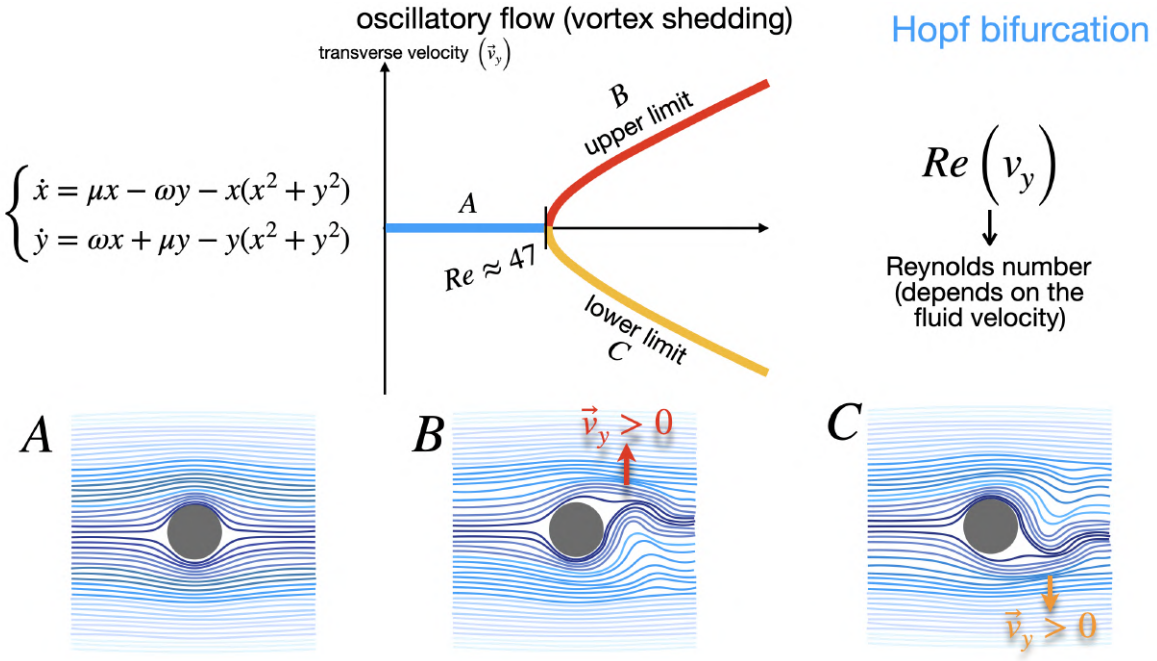


Imagine placing a cylinder on a fluid. Then you start to slowly turn up the speed of the fluid. At first the flow is smooth and steady, symmetrically wrapping around the cylinder without much fuss. It's stable.

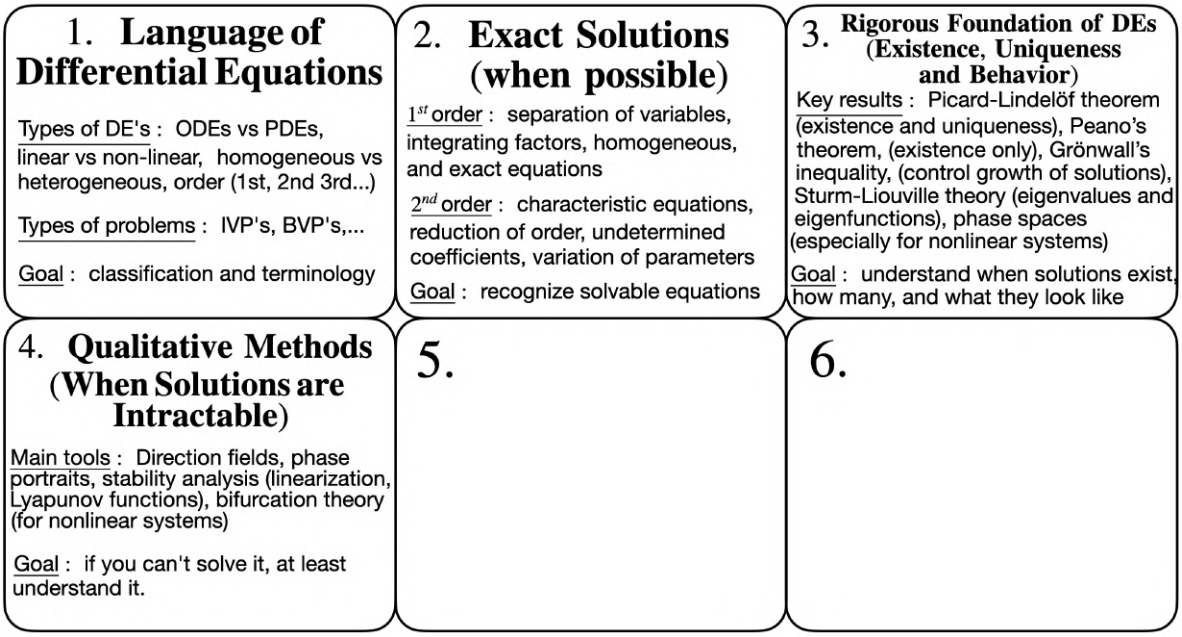


However, as the flow speed (that is tracked by the *Reynolds number*) crosses a critical limit, something changes. The system becomes unstable, with a phenomenon called vortex shedding.

The bifurcation appears in the graph at the point where the Reynolds number is about 47.

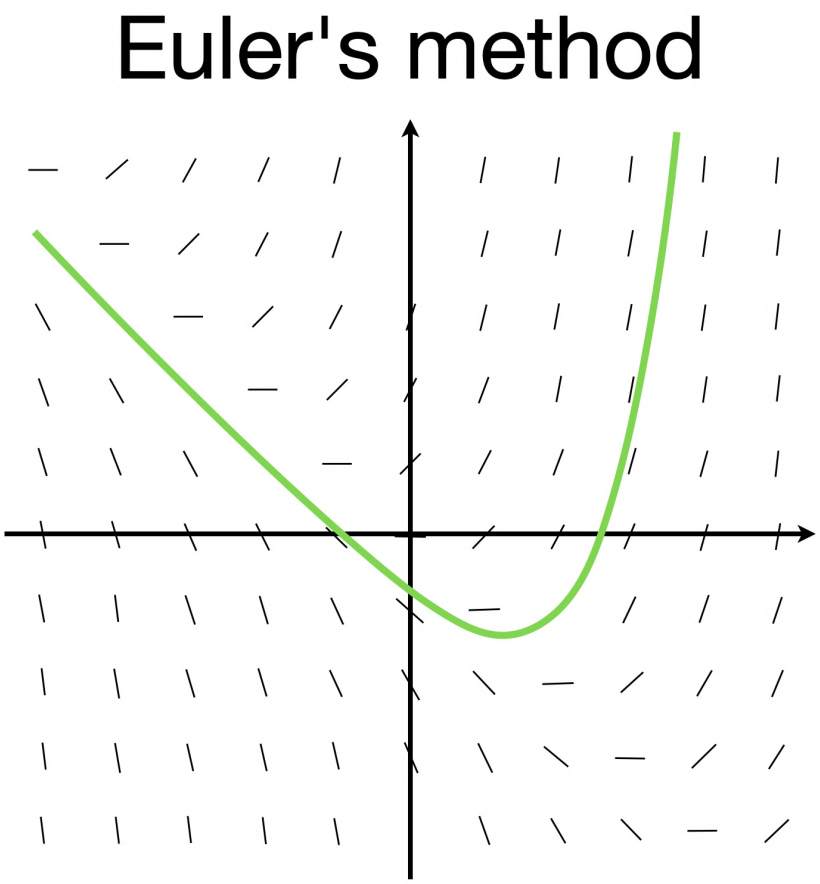


With this we conclude the fourth step to master DEs. We are ready to move on to the fifth.

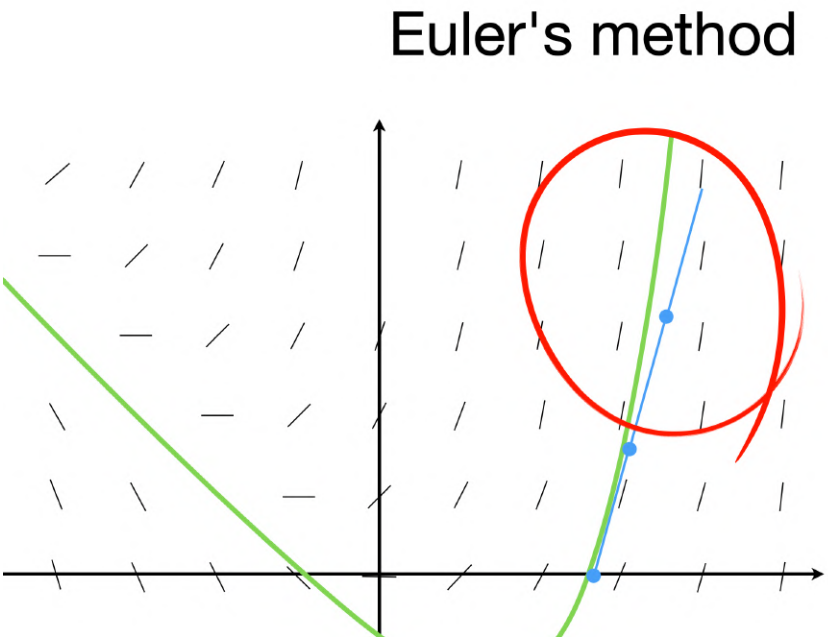


5 Numerical Methods

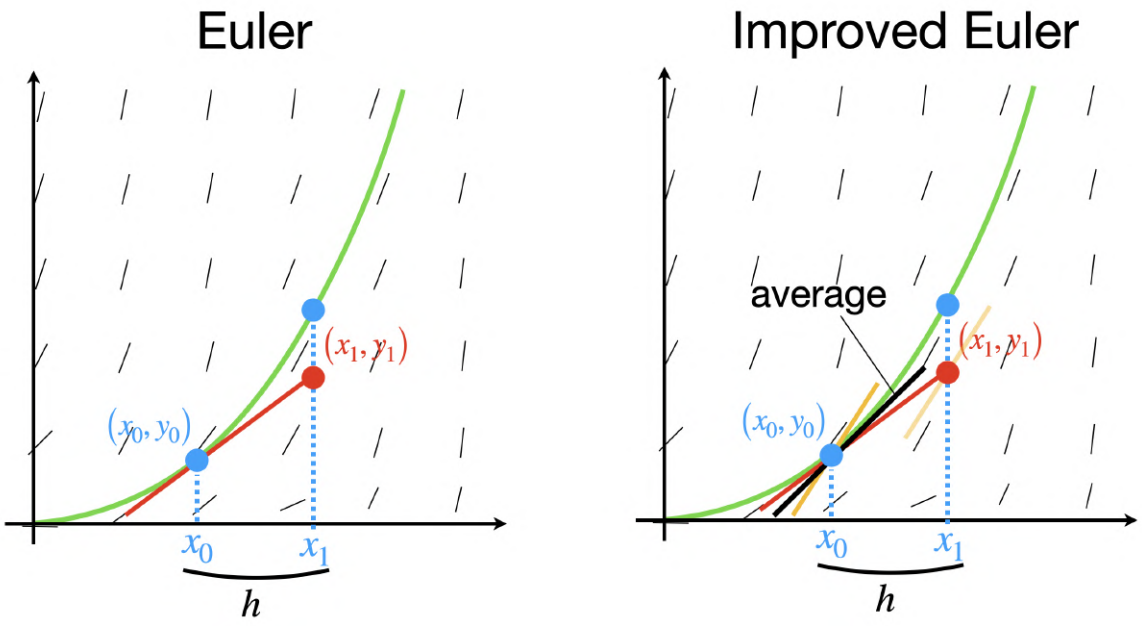
When DEs become too complex to solve analytically (especially for nonlinear systems), we turn to numerical methods.



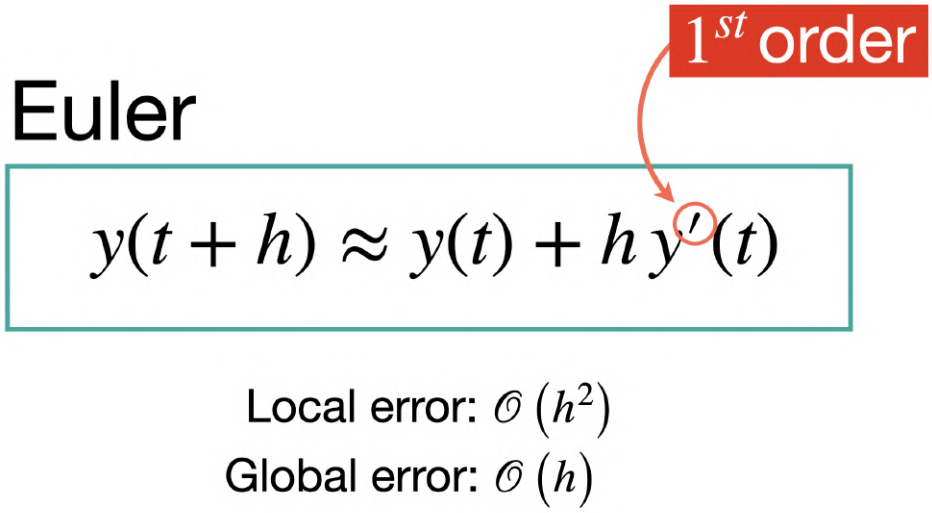
We use algorithms to compute approximate solutions at discrete points. One of the simplest is **Euler's method**, which takes tiny steps forward using the slope at each point to estimate solutions. When these steps tend to zero (in the limit), we end up with the "true" phase portrait of all solutions, since our approximation converges to the correct trajectories.



Euler's method is simple, but not very accurate, especially when using large step sizes, with errors that can quickly accumulate. So you can imagine how costly (from the computation point of view) it might be to have very small step sizes just to get a reasonable approximation. A more accurate method is the **Improved Euler Method**, which produces a better estimate by calculating the average slope over at each interval. Instead of just using the slope at the beginning of the step, it also includes the slope at the estimated end (and average of the two, actually). This produces a result that converges faster to the true solution.



Both (Euler and improved Euler) can be understood as approximations to the *Taylor series expansion* of the true solution $y(t)$. Euler's method corresponds to a first-order Taylor expansion, since it uses only the first derivative.



Improved Euler goes a step further by better approximating the slope over the interval, making it second-order accurate, and much closer to the actual trajectory.

Improved Euler

2^{nd} order

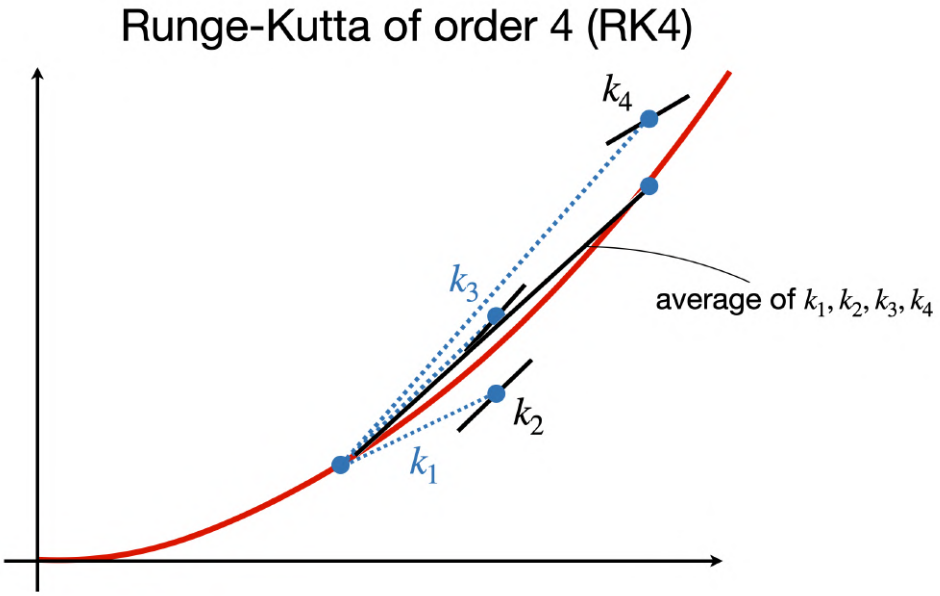
$$y(t+h) \approx y(t) + h\,y'(t) + \frac{h^2}{2} y''(t)$$

Local error: $\mathcal{O}\left(h^3\right)$

Global error: $\mathcal{O}\left(h^2\right)$

Now, when it comes to mastering the most powerful and widely used methods for numerically approximating solutions to DEs, you absolutely need to learn the **Runge-Kutta methods**, which are among the best.

These are a family of interactive techniques that build on the same core idea: using slopes to take steps. The most famous version, **Runge-Kutta of order 4** (**RK4**), combines multiple slope evaluations within each step to achieve fourth-order accuracy. This means the error decreases much faster as you shrink the step size, which gives highly accurate approximations without needing extremely small steps. In practice, **RK4** is usually considered the gold standard for solving ODEs numerically.



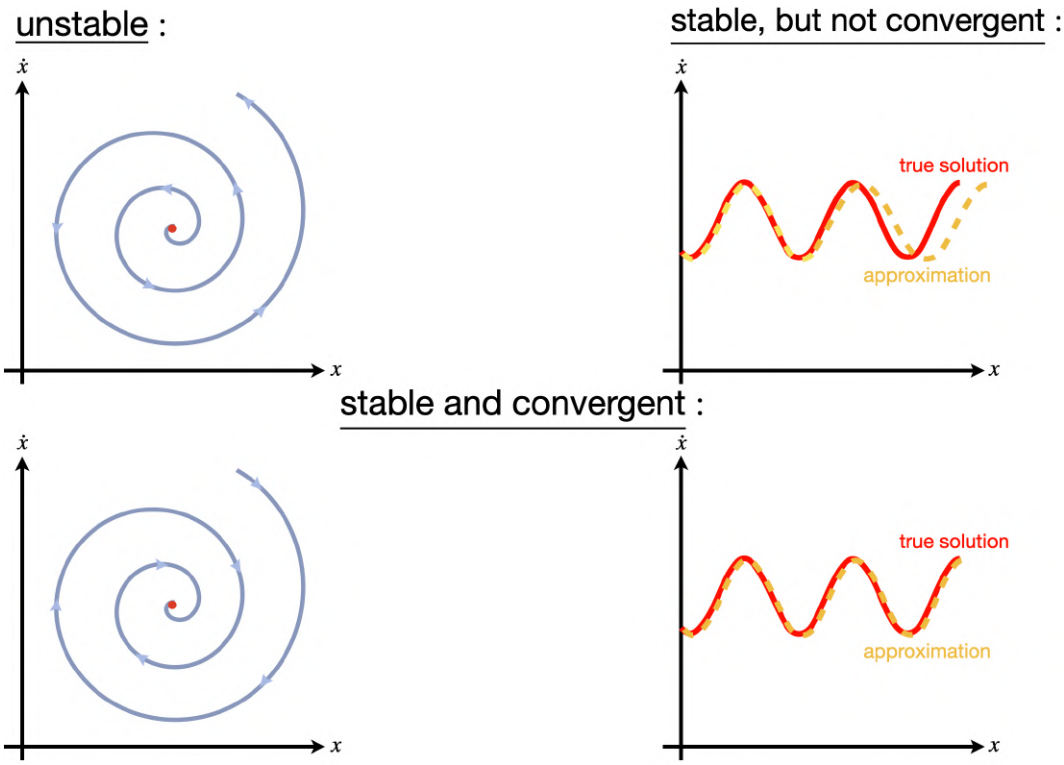
When working with PDEs, though, we often use *finite difference methods*, where we replace derivatives with differences across a grid in space and time.

Finite difference methods

Heat equation

$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \rightarrow \frac{u_i^{n+1} - u_i^n}{\Delta t} = \alpha \frac{u_{i+1}^n - 2\,u_i^n + u_{i-1}^n}{\Delta x^2}$$

However, just getting an answer is not enough. We also need to know if we can trust it. That's where concepts like **stability**, **convergence**, and **error analysis** come in. They help us ensure that as we refine our steps (making them smaller), our solution behaves consistently, getting closer to the true solution, rather than spiraling out of control because of numerical errors.

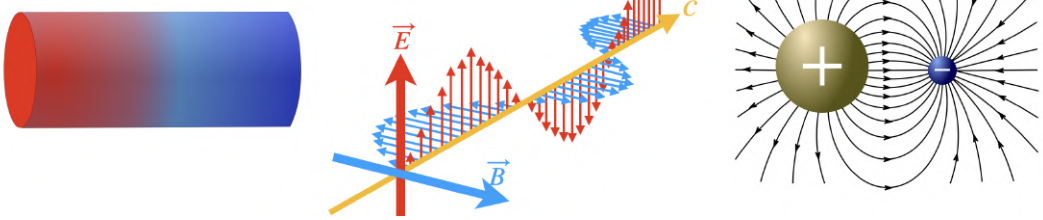


This concludes our fifth step (numerical methods), and we are ready to move on to the last step!

1. **Language of Differential Equations**	2. **Exact Solutions (when possible)**	3. **Rigorous Foundation of DEs (Existence, Uniqueness and Behavior)**
Types of DE's : ODEs vs PDEs, linear vs non-linear, homogeneous vs heterogeneous, order (1st, 2nd 3rd...) Types of problems : IVP's, BVP's,... Goal : classification and terminology	1^{st} order : separation of variables, integrating factors, homogeneous, and exact equations 2^{nd} order : characteristic equations, reduction of order, undetermined coefficients, variation of parameters Goal : recognize solvable equations	Key results : Picard-Lindelöf theorem (existence and uniqueness), Peano's theorem, (existence only), Grönwall's inequality, (control growth of solutions), Sturm-Liouville theory (eigenvalues and eigenfunctions), phase spaces (especially for nonlinear systems) Goal : understand when solutions exist, how many, and what they look like
4. **Qualitative Methods (When Solutions are Intractable)**	5. **Numerical Methods**	6.
Main tools : Direction fields, phase portraits, stability analysis (linearization, Lyapunov functions), bifurcation theory (for nonlinear systems) Goal : if you can't solve it, at least understand it.	Main : Euler's method, Improved Euler, Runge-Kutta, finite-difference methods (for PDEs), stability, convergence, and error analysis Goal : approximate solutions correctly	

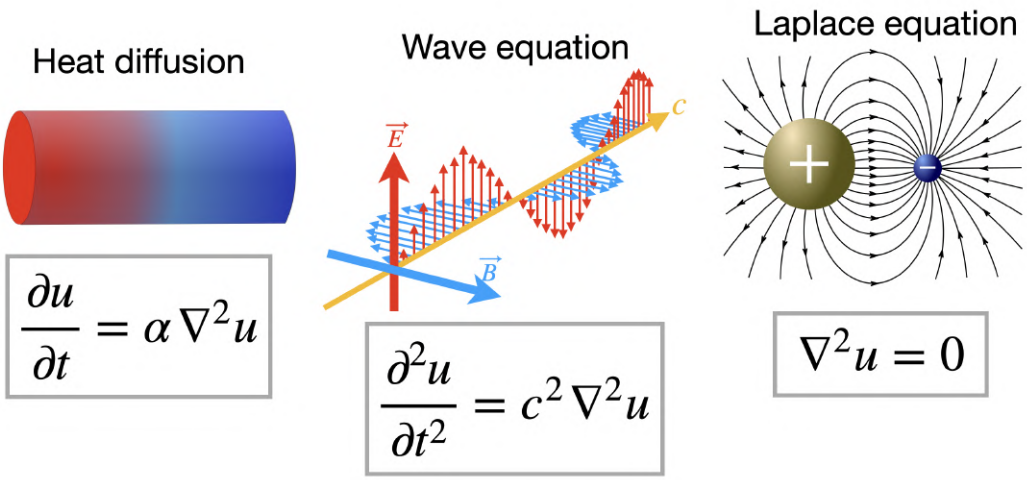
6 PDEs & Mathematical Physics

PDEs are super important in Mathematical Physics. They describe how physical quantities like **heat**, **wave displacement**, or **electrostatic potential** evolve over both space and time.



Some of the so-called **canonical equations** include the Heat diffusion (modeling heat flow), the Wave equation (modeling vibrations), and the Laplace equation (modeling steady-state potentials).

Canonical Equations



These equations describe continuous systems involving multiple interacting variables, and that's the main difference compared to ODEs, which involve only one independent variable and its derivatives.

In mathematical physics, it's striking how the vast majority of fundamental laws are described by continuous equations, often in the form of DEs. This reflects our intuitive assumption that physical quantities (such as temperature, displacement, electric potential, or probability amplitude) vary smoothly across space and time. As a result, DEs naturally arise as the mathematical description of continuous change.

However, not all of nature is continuous. Some phenomena are inherently discrete, especially in QM. In such cases, the governing laws often involve difference equations or operators with discrete spectra. A classic example is the time-independent Schrödinger equation:

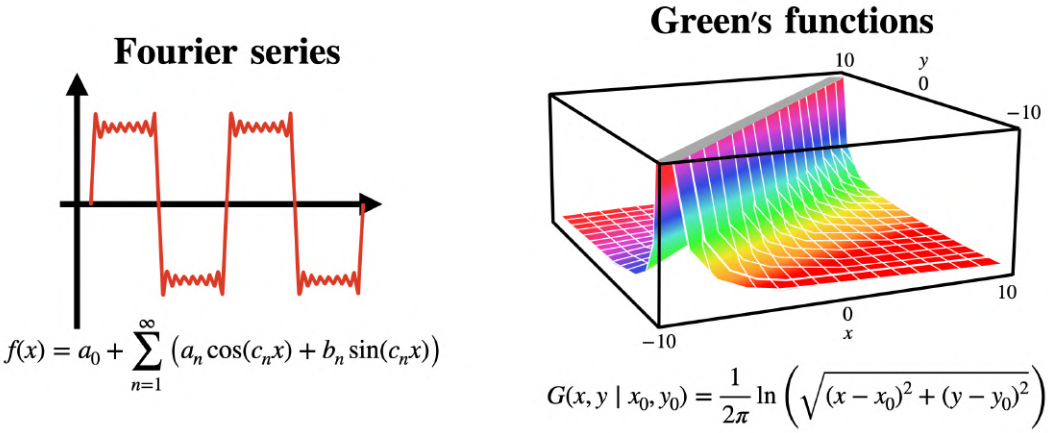
$$\hat{H}\,\psi = E\,\psi \tag{1}$$

Where $\hat{H}$ is the **Hamiltonian operator**. This is a DE, sure, but its solutions (such as the energy levels of the hydrogen atom) are *quantized*, i.e. discrete:

$$E_n = -\frac{13.6\,\text{eV}}{n^2}, \quad n = 1, 2, 3, \ldots \tag{2}$$

So, even though continuous processes described by DEs definitely dominate physical theories, discrete mathematics governs the quantum world and, in doing so, it also governs the very foundations of nature.

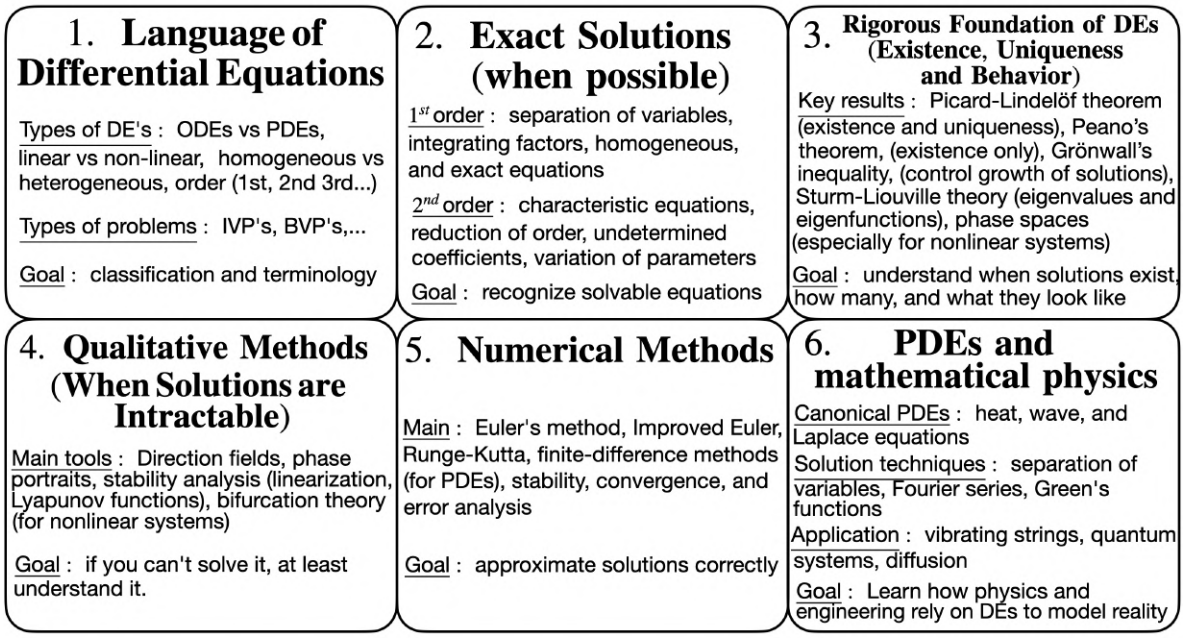
To solve PDEs, mathematicians and physicists use analytical techniques such as **separation of variables** (which we've already seen earlier), **Fourier series** (which express complicated functions as sums of sines and cosines, so that it's easier to handle boundary conditions in periodic problems), and also we have **Green's functions** (which describe the response of a system to a point source).



The thing that all of these methods have in common is the goal of simplifying complex systems by breaking them into smaller, and more manageable, parts. And the overall goal of PDEs in mathematical physics is to understand how these equations encode physical laws, and as a consequence, to use them to predict future behaviors of physical systems.

Conclusion

Therefore, these are the 6 steps to master DEs:



Practice (Exercises)

We added one exercise for each of the 6 steps (with the detailed solutions). Also, I remind you guys that we provide the FULL-PDF version for free for all members of the channel. Just join us on YouTube! We'd like to keep our videos free of interruptions and sponsors, so that the sole focus is the subject at hand. But in order to do that we need your help. Thanks for supporting our work!

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