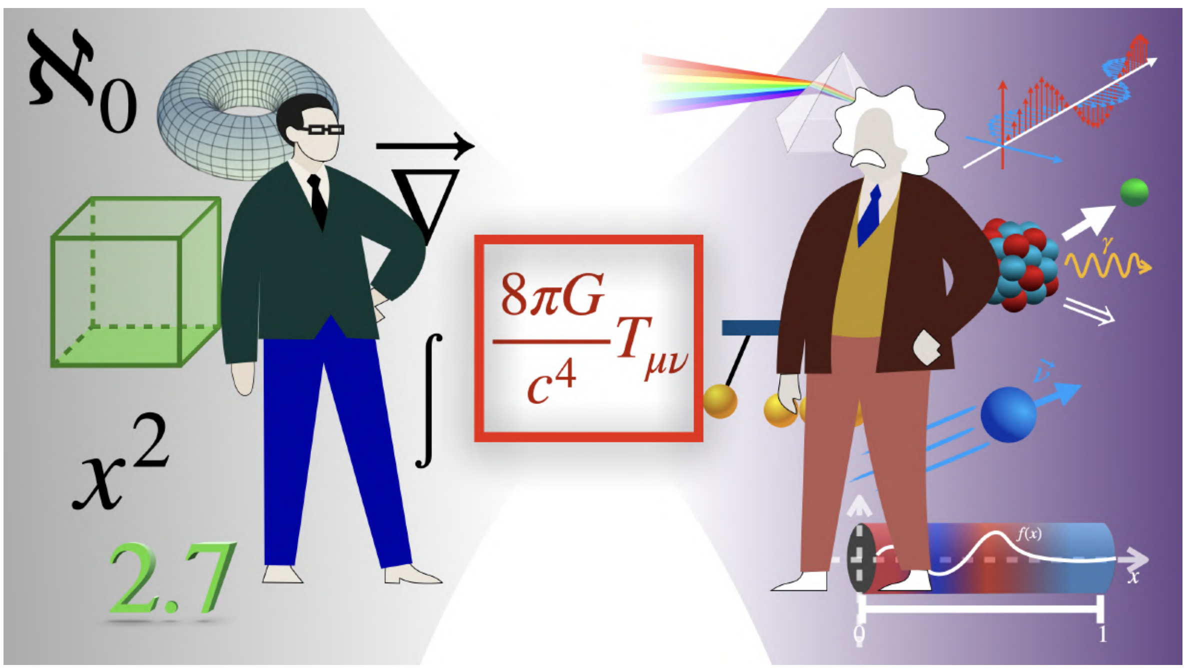
# What’s on the Right Side of Einstein’s Equation? (Math vs Physics)

by DiBeos



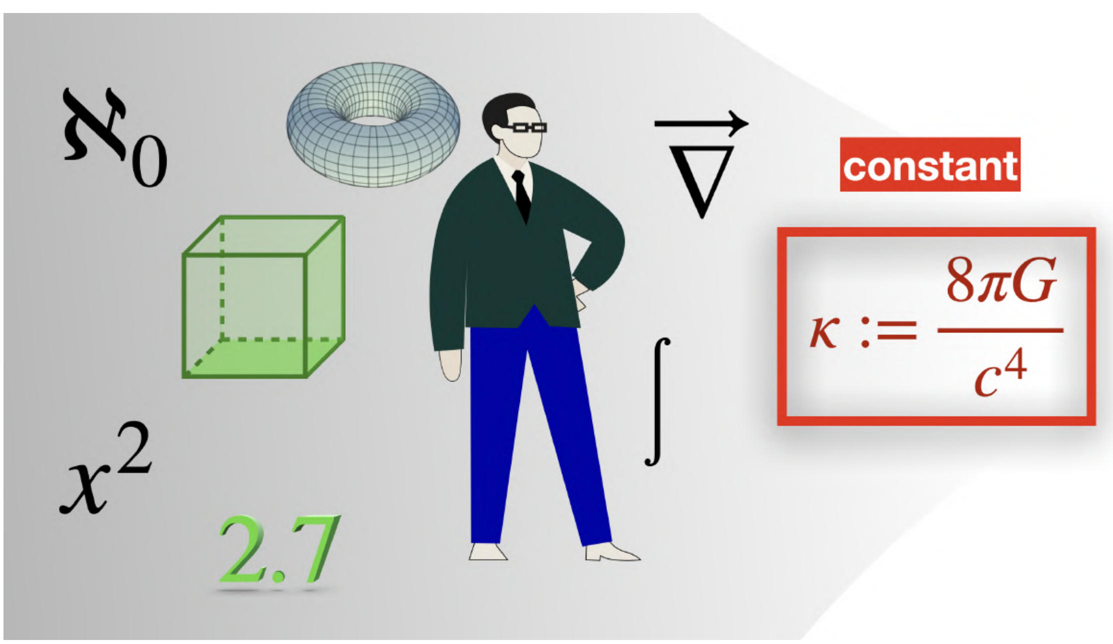

*"In geometric and physical applications, it always turns out that a quantity is characterized not only by its tensor order, but also by symmetry."*
**– Hermann Weyl**

# Introduction

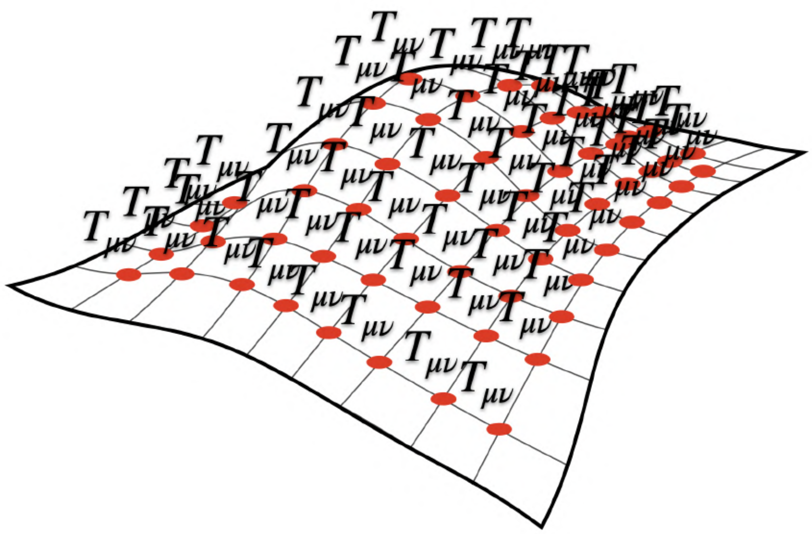
When looking at the RHS (:= right-hand side) of Einstein's field equations, mathematicians and physicists see different things from each other:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \boxed{\frac{8\pi G}{c^4} T_{\mu\nu}}$$

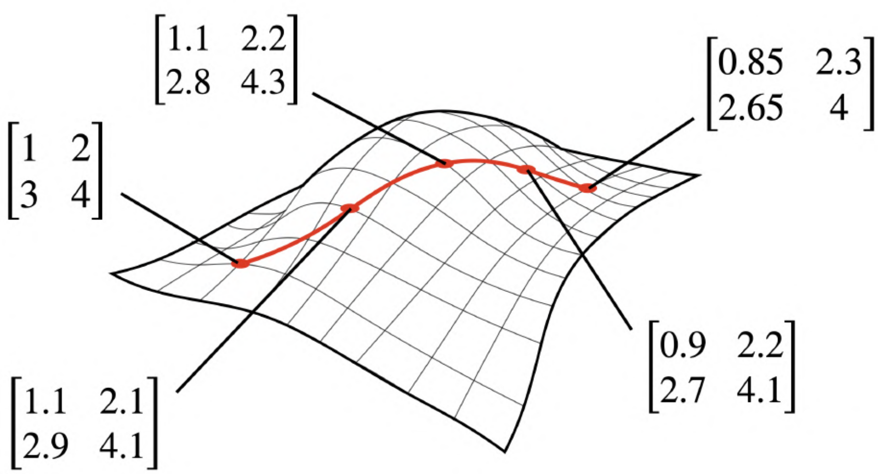
For the mathematician, this constant $\left(\frac{8\pi G}{c^4}\right)$ is just that, a constant. He's not interested in the fact that it involves two of the most fundamental quantities in nature (i.e. the *gravitational constant* $G = 6.67 \times 10^{-11}\ N \cdot m^2/kg^2$ and the *speed of light* $c = 299\,792\,458\ m/s$). So the mathematician sees it as nothing but a scaling factor that might as well be called $\kappa = \frac{8\pi G}{c^4}$, for example.



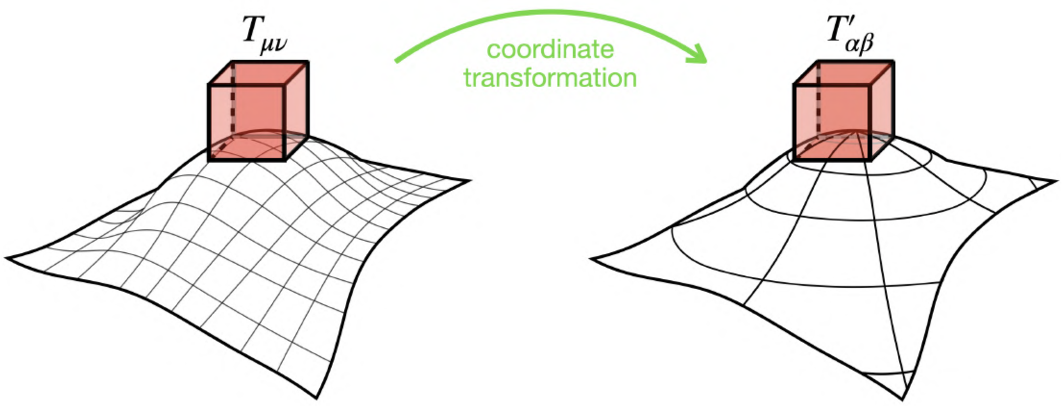
This tensor $\left(T_{\mu\nu}\right)$ is described as a rank-2 symmetric object (2 indices) defined on a differentiable manifold.



It is defined pointwise. Its components vary smoothly with the coordinates.

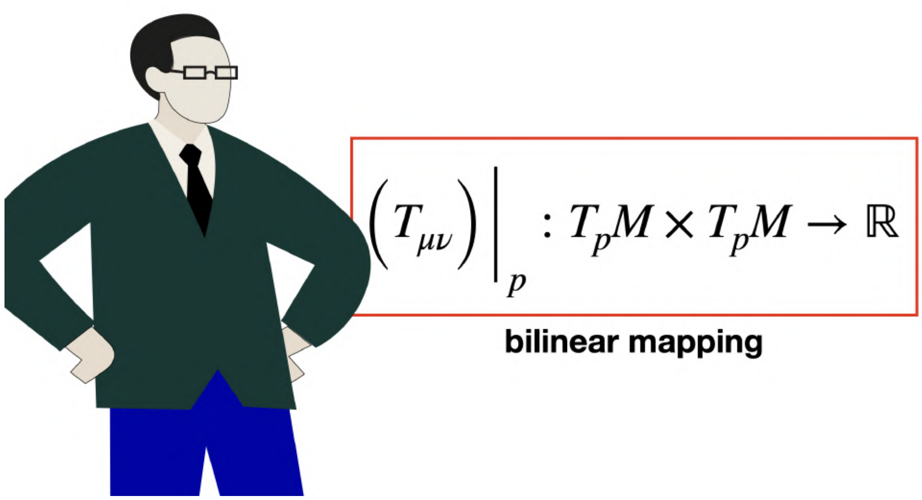


And it is covariant under coordinate transformation.

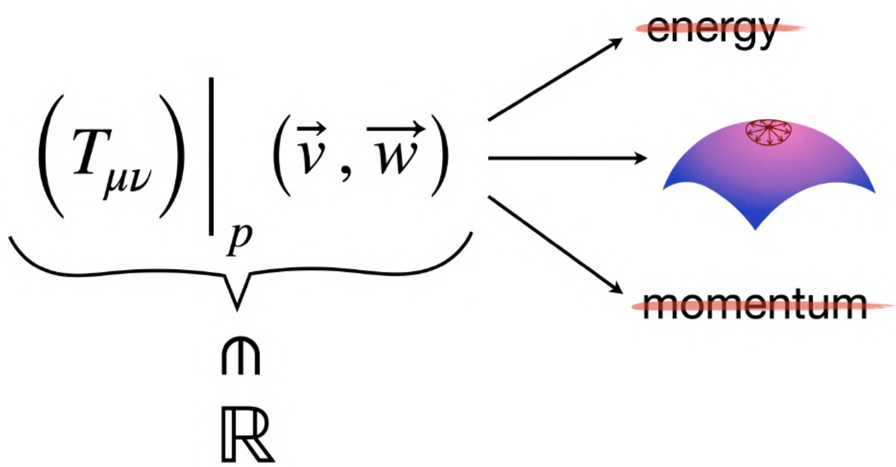



It does not depend on the local coordinate choice, it is an intrinsic geometric object.

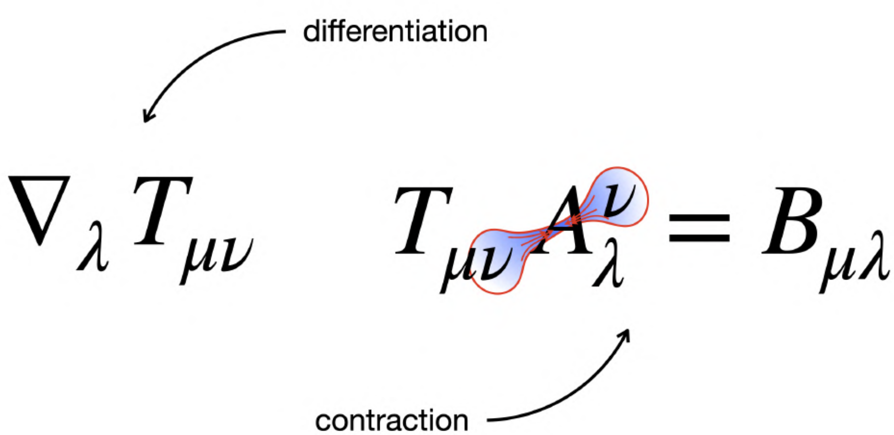
The mathematician would also describe this tensor as a bilinear mapping, so a function that takes 2 tangent vectors at a point on the manifold and returns a single real number.



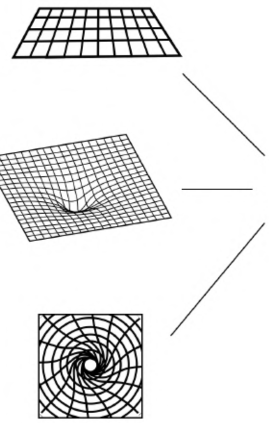
This real number (the output of the bilinear map) is not interpreted as energy or momentum, as a physicist would, but instead it's viewed mathematically as encoding a relation between directions on the manifold at that point.




The mapping is smooth, and so we can differentiate it $\left(\nabla_\lambda T_{\mu\nu}\right)$ and contract it with other tensors $\left(T_{\mu\nu}A^\nu_\lambda = B_{\mu\lambda}\right)$.



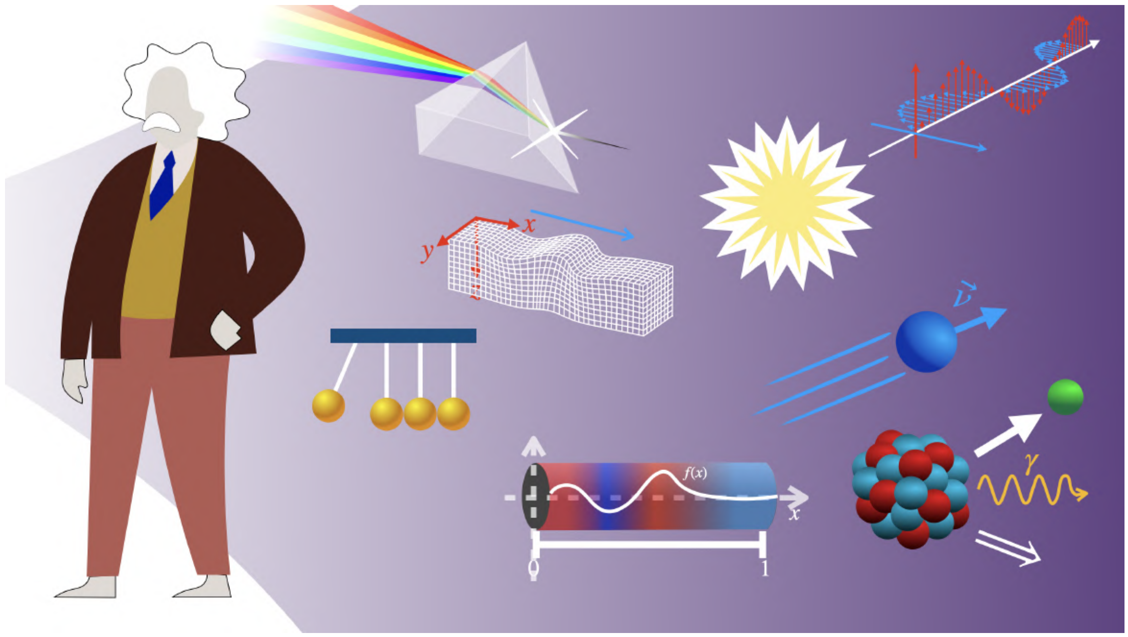
Each one of the operations would produce a different result that directly influences the LHS (:= left-hand side) of Einstein's equation, i.e. the geometry and curvature of spacetime.



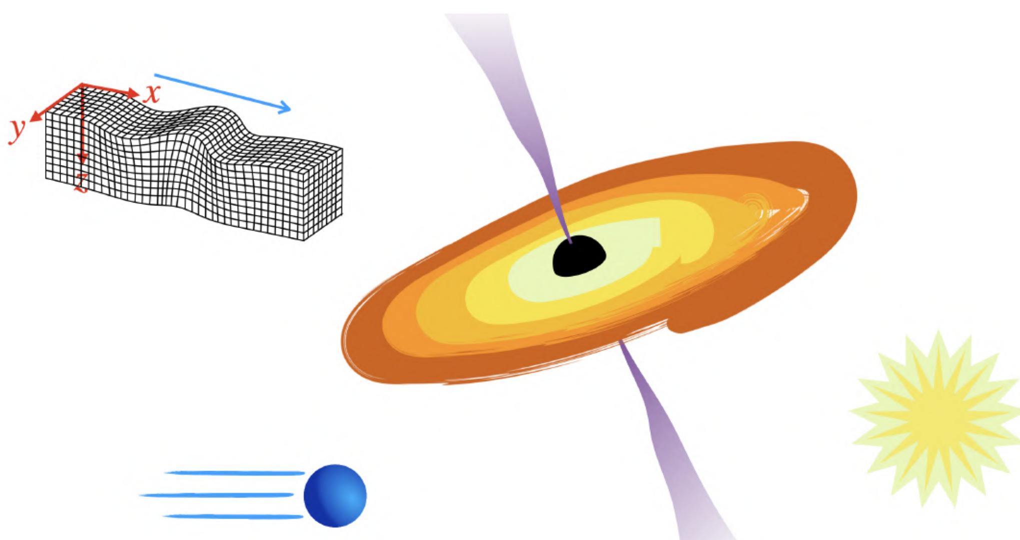
LHS RHS

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

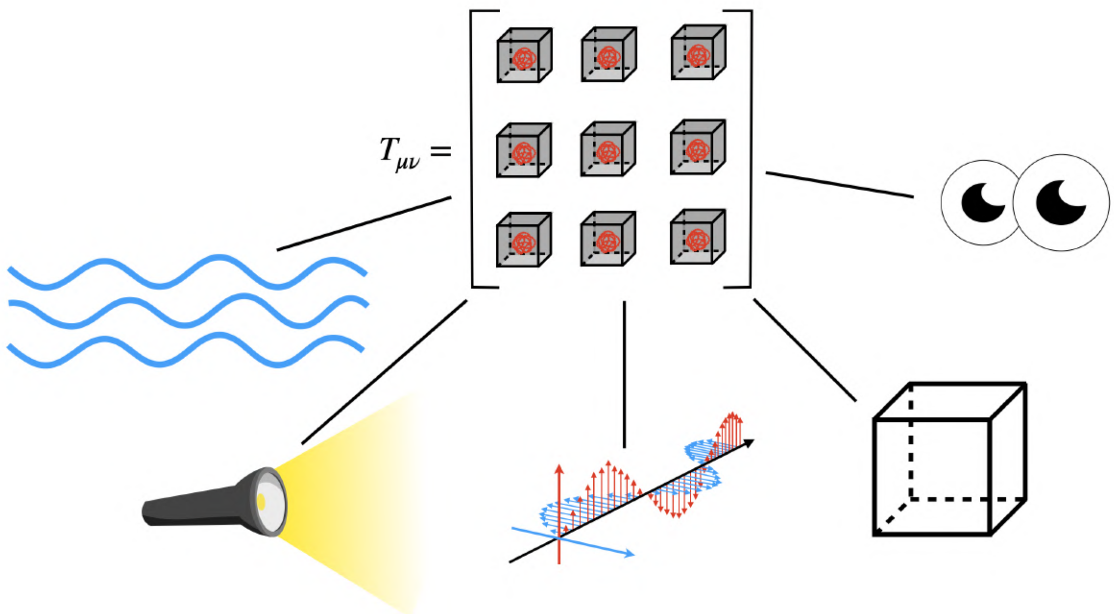
Now, the physicist would see the situation from a different perspective...



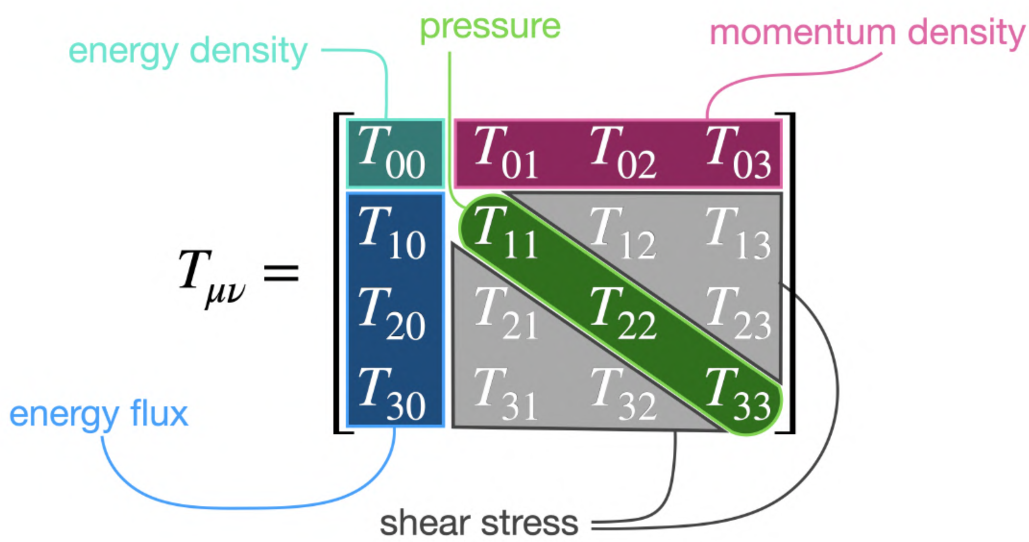

Each component represents something tangible: *energy*, *momentum*, *pressure*, *radiation*... It tells you how matter moves, how it pushes and pulls, and how it exchanges energy.



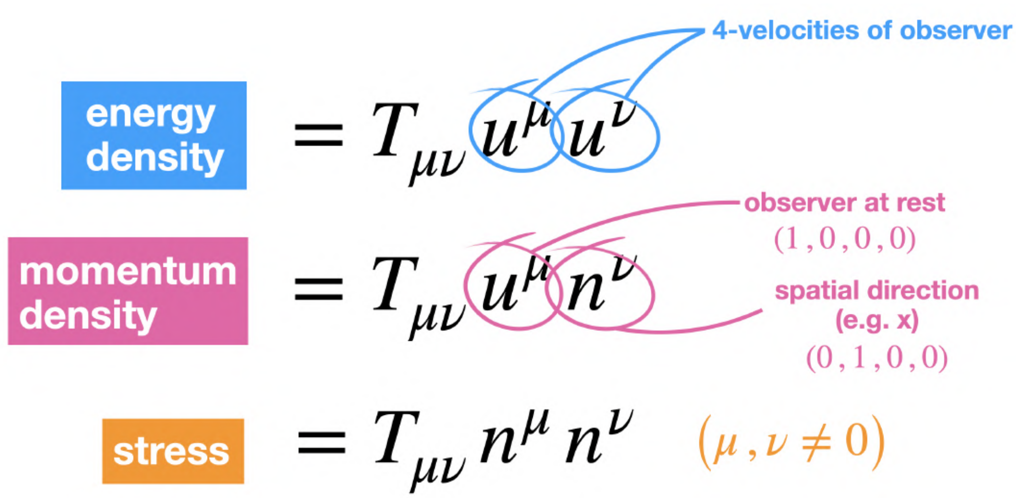
$T_{\mu\nu}$ encodes the state of a fluid, a beam of light, an electromagnetic field, or even the vacuum, but always in terms of observables, measurable quantities.




That's a very common misconception: that only matter is capable of bending spacetime and producing curvature. This tensor, $T_{\mu\nu}$, tells us a different story: **stress** (which is a generalization of pressure, including shear and tension), **energy** (in the form of matter or radiation) and **momentum** (that is, energy in motion or flow across space) are all capable of producing spacetime curvature.

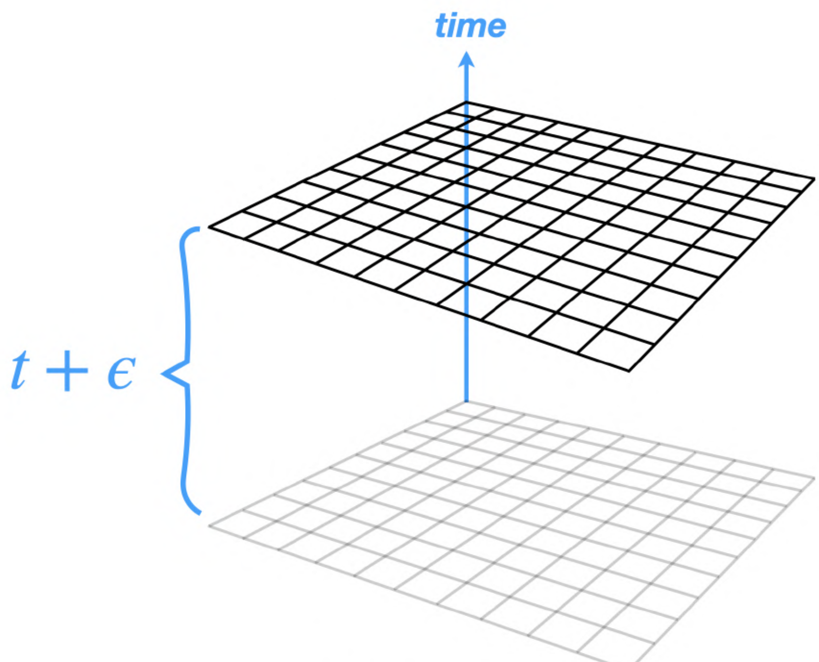



Interestingly, this matrix encodes all this information, and if at any moment you want to extract a piece of data from it, all you gotta do is contract it with the appropriate basis vectors or tensors corresponding to the physical quantity you're interested in:



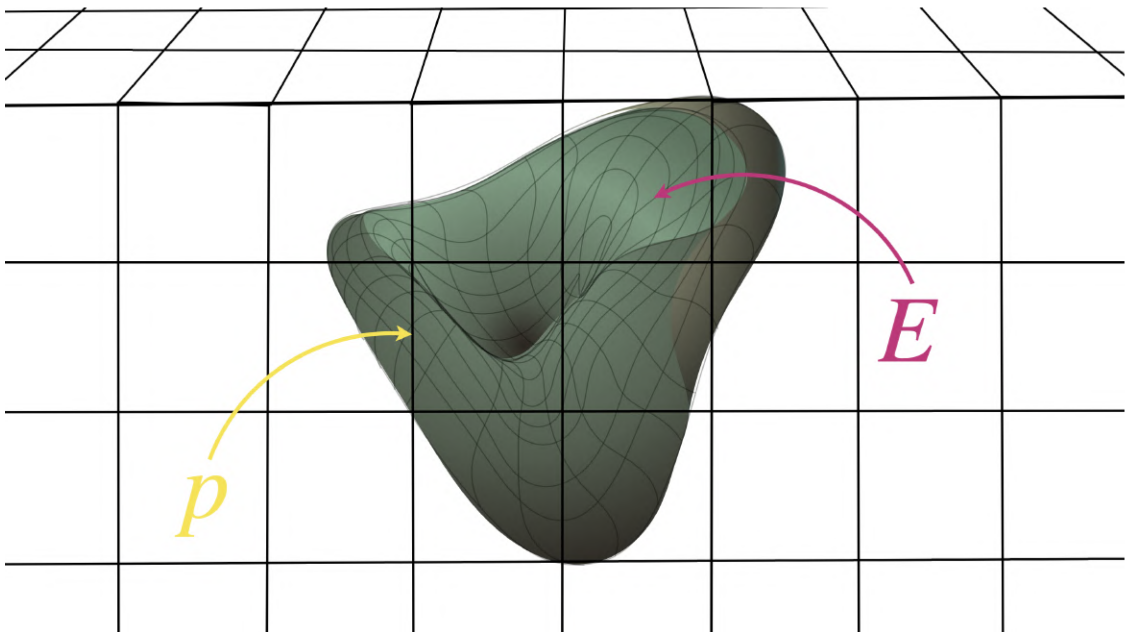

One of the most elegant bridges between pure mathematics and physics comes from a result called **Noether's theorem**. It tells us that whenever a system has a continuous symmetry, there is a conserved quantity.

In the context of Einstein's equation, if the laws of physics (and more importantly, the geometry itself) remain unchanged when you shift the coordinate system slightly in space or time, that's called translational symmetry.

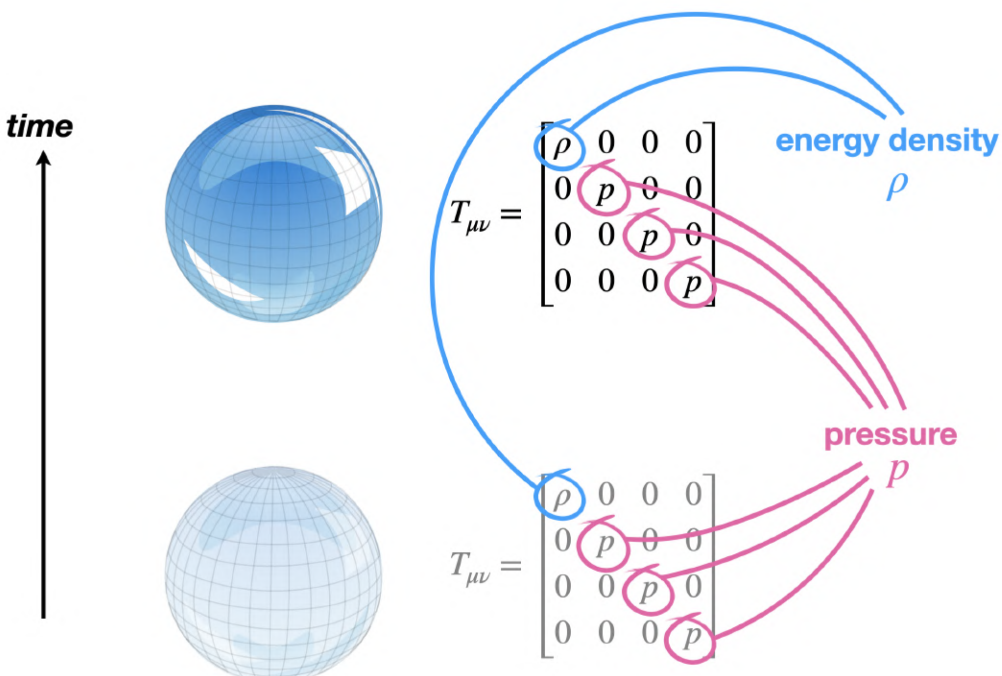



translational symmetry

From this symmetry, *Noether's theorem* gives us conservation of energy and momentum, which means, for example, that the total energy inside a closed region of space doesn't randomly disappear as you move forward in time, and momentum doesn't magically appear or vanish as you move through space.

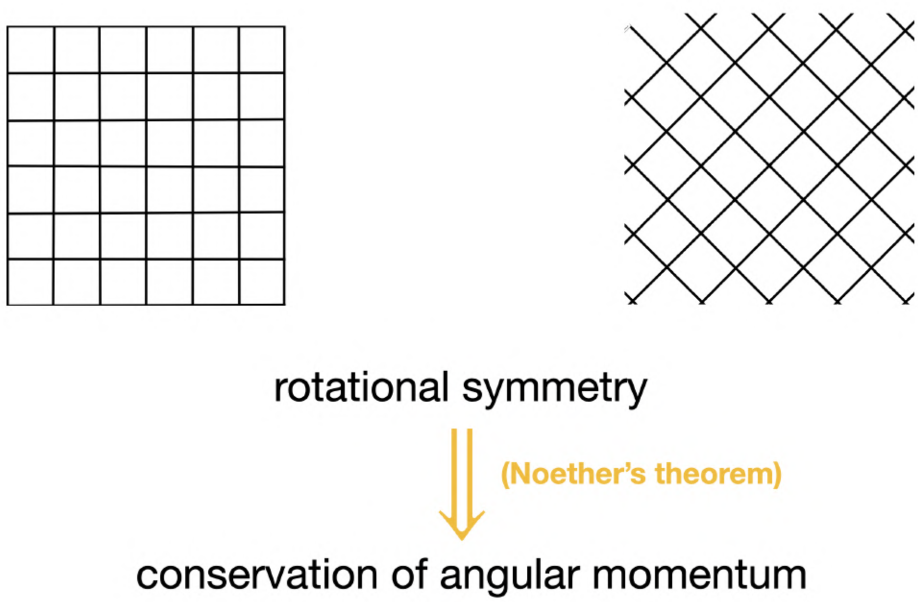



As a concrete example, imagine a perfect fluid at rest. This (below) is a volume of space that looks exactly the same everywhere and in every direction. There's no motion in space, but there's always motion in time.

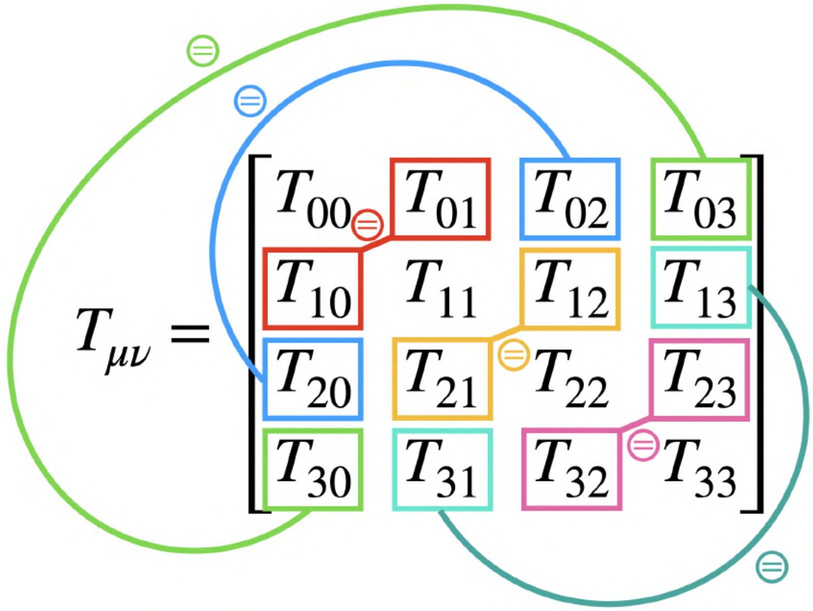


The energy density $\rho$ is constant. The pressure $p$ is the same in all spatial directions. This uniformity gives us something very powerful: **translational symmetry**. According to Noether's theorem, this symmetry leads to *conservation of energy and momentum*.

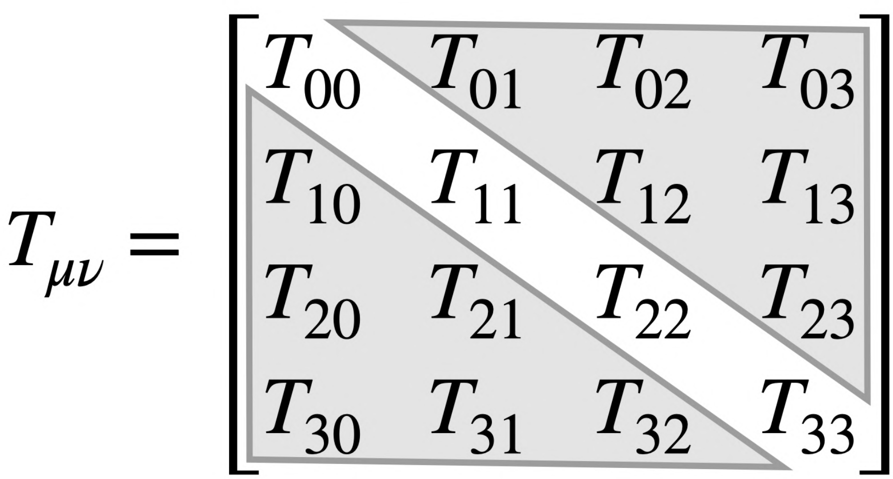
But the story doesn't stop there. Imagine rotating your coordinate system (like spinning your frame of reference). If the geometry and physical laws remain unchanged under such a rotation, that's **rotational symmetry**. Noether's theorem tells us that this symmetry leads to the *conservation of angular momentum*.




Mathematically, this conservation law emerges when the tensor is symmetric: $T_{\mu\nu} = T_{\nu\mu}$.



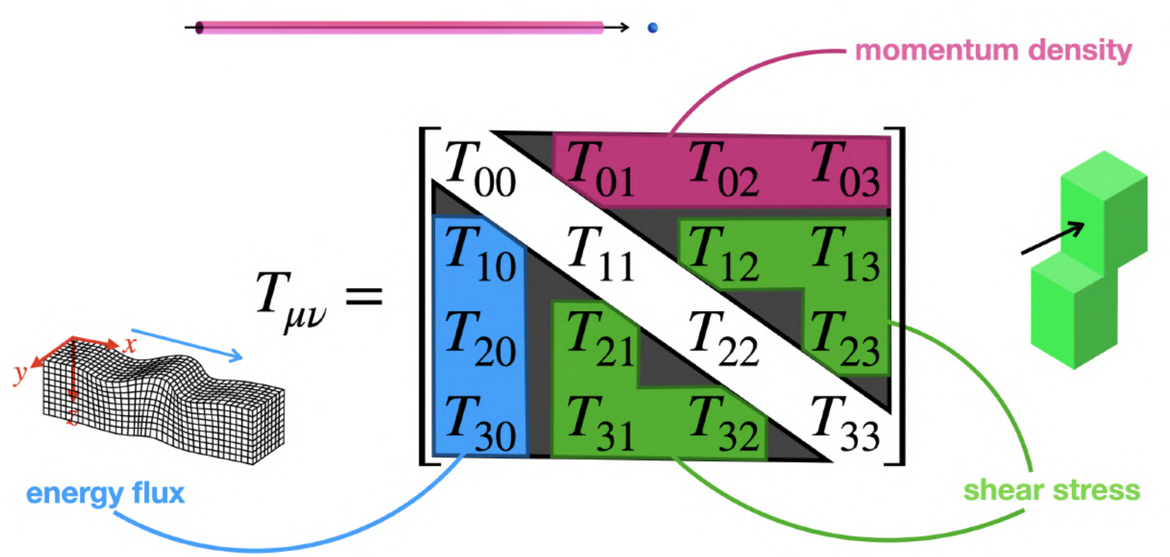

$T_{01} = T_{10}$, $T_{02} = T_{20}$, and so on. Basically, these triangular regions (below) are the same.




The components encode information about *shear stress*, which tells us how the motion in one direction affects neighboring layers.

Other components represent *momentum density* in each direction. Think of it as the amount of "push" (from either matter or radiation) contained within a tiny volume of space, and in a specific direction.

And others give us the *energy flux*, i.e. how much energy is moving through space, in each direction.

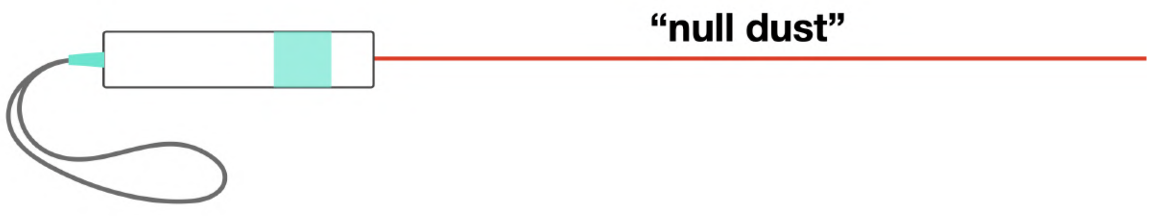



Notice that for us to have this kind of symmetry (i.e. rotational symmetry), the momentum density must be equal to the energy flux. It means that the amount of energy flowing in a spatial direction must be the same as the amount of momentum stored in that direction over time. This balance is what ensures angular momentum conservation.

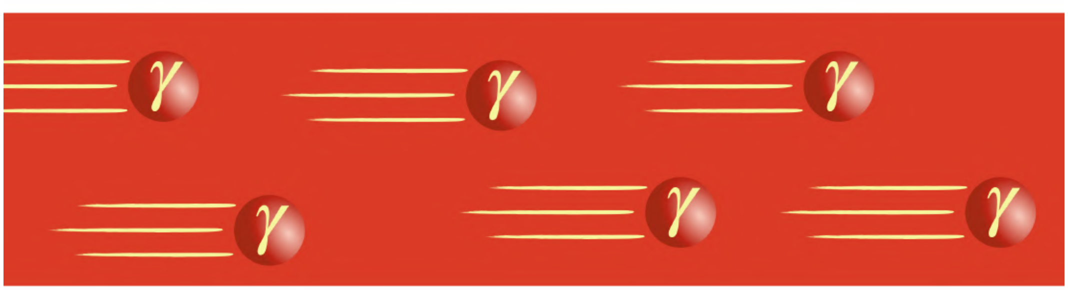
And lastly, in General Relativity, we deal with curved spacetime, which means we must go beyond flat translations and rotations. Here, the symmetry is much deeper: it's called **general covariance**.

Indeed, Einstein has originally named his work "*Theory of Invariance*", referring to this type of symmetry, also known as **diffeomorphism invariance**. It means that the equations of physics (and the geometry itself) look the same in any coordinate system you choose. This symmetry doesn't lead to a global conservation law, but to a local one.

Mathematically, it tells us that this tensor must satisfy a sort of geometric divergence-free condition, expressed as $\nabla^\mu T_{\mu\nu} = 0$. Intuitively, it means that energy and momentum are conserved locally, from point to point, even when the manifold is curved.




More concretely, imagine a narrow beam of light traveling in the $x$-direction, like a laser. This is a classic example of what's called a "*null dust*": a stream of massless particles (such as photons) all moving together at the speed of light.




The tensor for this system stores information about how energy and momentum are distributed and how they flow through spacetime.


$$T_{\mu\nu} = \begin{bmatrix} T_{00} & T_{01} & 0 & 0 \\ T_{10} & T_{11} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$T_{00}$ is the energy density present at that point in space.

$T_{01}$ is the momentum density in the $x$-direction.

$T_{10}$ is the energy flux.

And $T_{11}$, the pressure in the $x$-direction, so the amount of force per unit area being exerted on a surface perpendicular to the $x$-axis.

All other components are zero, since the light beam is not moving along the $y$- or $z$-directions, and there is no pressure or shear in those directions anyway.

In flat space, for example, the local conservation condition becomes:

$$\boxed{\nabla^\mu T_{\mu\nu} = 0} \xrightarrow{\text{(flat spacetime)}} \partial^\mu T_{\mu\nu} = 0 \implies$$

$$\implies \partial^0 T_{0\nu} + \partial^1 T_{1\nu} + \cancelto{0}{\partial^2 T_{2\nu}} + \cancelto{0}{\partial^3 T_{3\nu}} = 0 \implies$$

$$\implies \quad \eta^{0\lambda}\partial_\lambda T_{0\nu} + \eta^{1\lambda}\partial_\lambda T_{1\nu} = 0 \;\text{, where } \eta^{\sigma\lambda} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \implies$$

$$\implies \quad \boxed{-\frac{\partial}{\partial x^0} T_{0\nu} + \frac{\partial}{\partial x^1} T_{1\nu} = 0}$$

<u>For $\nu = 0$</u>: $\left(x^0 = t \;;\; x^1 = x\right)$

$$\boxed{-\frac{\partial}{\partial t} T_{00} + \frac{\partial}{\partial x} T_{10} = 0} \quad \implies$$

$\implies$ **Local Conservation of Energy (continuity equation for energy).**

Here, $T_{00}$ is **energy density**, and $T_{10}$ is **energy flux** in the $x$-direction.

<u>For $\nu = 1$</u>:

$$\boxed{-\frac{\partial}{\partial t} T_{01} + \frac{\partial}{\partial x} T_{11} = 0} \quad \implies \quad \textbf{Momentum Conservation.}$$

Where $T_{01}$ is **momentum density (in the $x$-direction)**, and $T_{11}$ is **pressure**.

<u>For $\nu = 2$ or $\nu = 3$</u>:

$$\boxed{-\frac{\partial}{\partial t} \cancelto{0}{T_{02}} + \frac{\partial}{\partial x} \cancelto{0}{T_{12}} = 0} \qquad \boxed{-\frac{\partial}{\partial t} \cancelto{0}{T_{03}} + \frac{\partial}{\partial x} \cancelto{0}{T_{13}} = 0}$$

$\therefore \nexists$ flow of *energy*, *stress*, or *momentum* in the $y$- and $z$-directions.

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