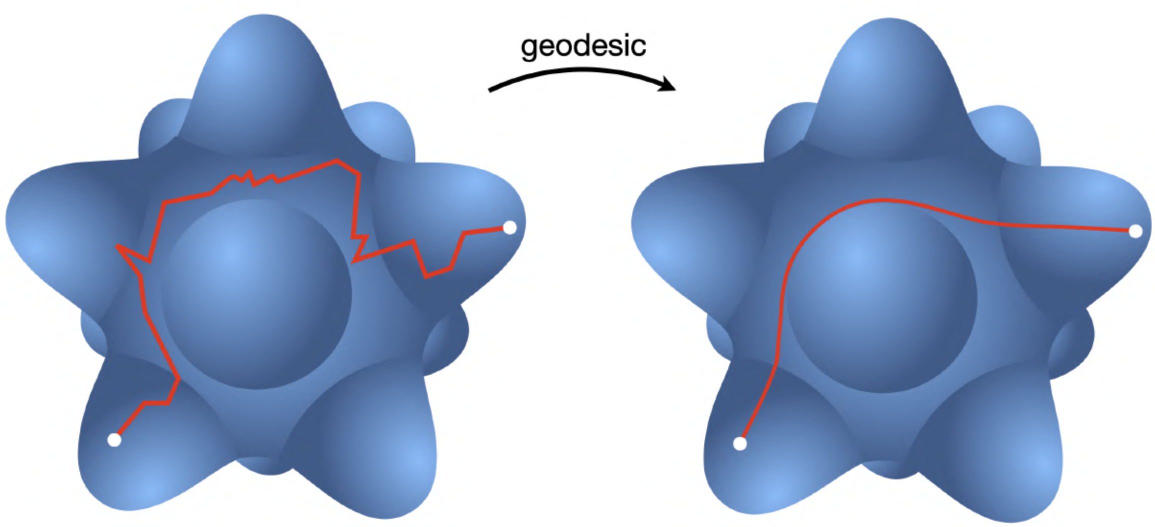
Top 5 Methods for Solving the Geodesic Equation

by DiBeos



"I often say that when you can measure what you are speaking about, and express it in numbers, you know something about it; but when you cannot measure it, when you cannot express it in numbers, your knowledge is of a meagre and unsatisfactory kind." – **Lord Kelvin**

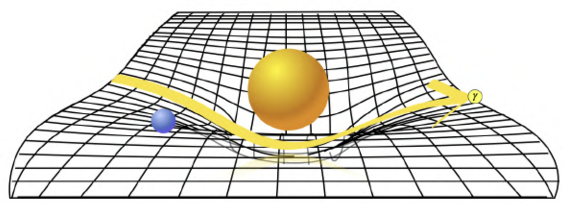
Introduction

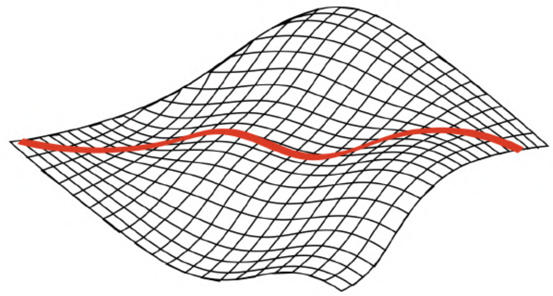
1. **Symmetries & Killing Vectors**
2. **First Integrals / Conservation Laws**
3. **Separation of Variables**
4. **Coordinate Transformations**
5. **Integrability & Quadratures**

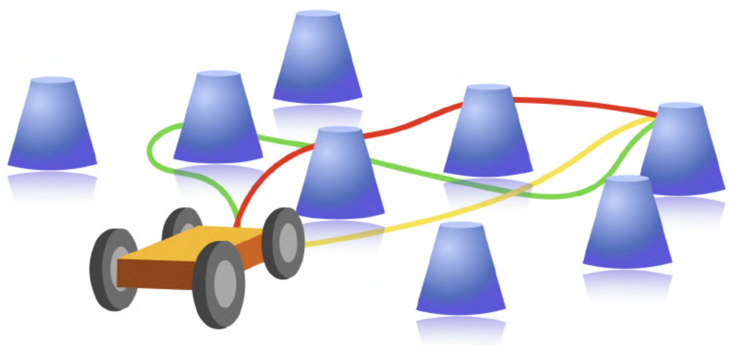
This (above) is the list of the best analytical methods for solving the geodesic equation.

In **Pure Mathematics**, the geodesic equation is one of the core concepts in Differential Geometry. It encodes the notion of "the straightest possible path" on curved spaces, and without it, it's impossible to truly understand the geometry of any particular manifold.

In **Applied Mathematics**, its power is even more evident. For example, it describes particle trajectories in General Relativity, light propagation in optics, and optimal paths in navigation and robotics.







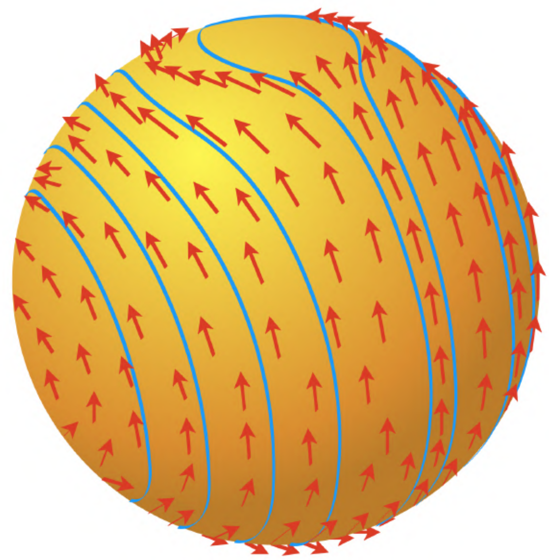
That said, solving it exactly is usually really tough, but when the geometry allows, we can use some very powerful analytical tools that make the problem much simpler.

$$\boxed{\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j = 0}$$

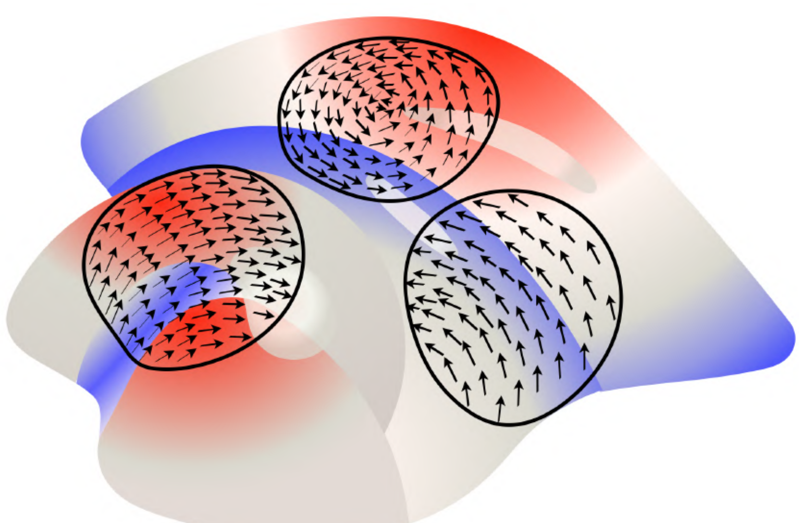
Let's see the first of the top 5 analytical methods for reducing its complexity and find exact solutions:

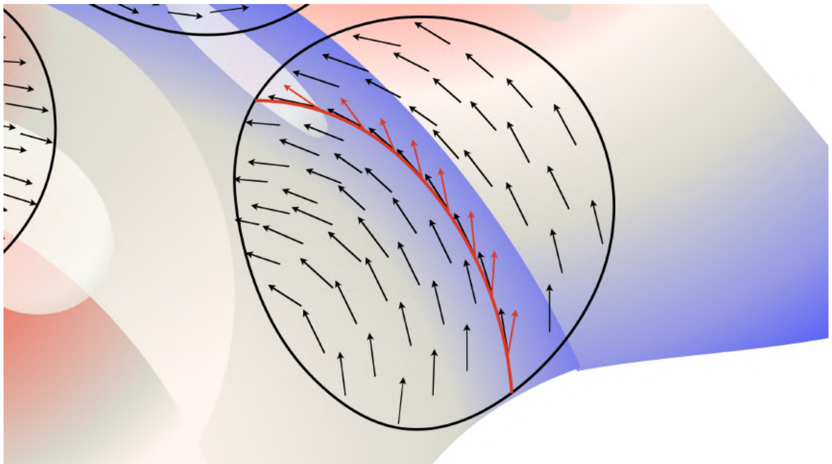
1 Symmetries & Killing Vectors

As the name gives away, this method explores symmetries in the geodesic equation and in the space you're studying, and it does so by using very important mathematical objects called *Killing vectors*.



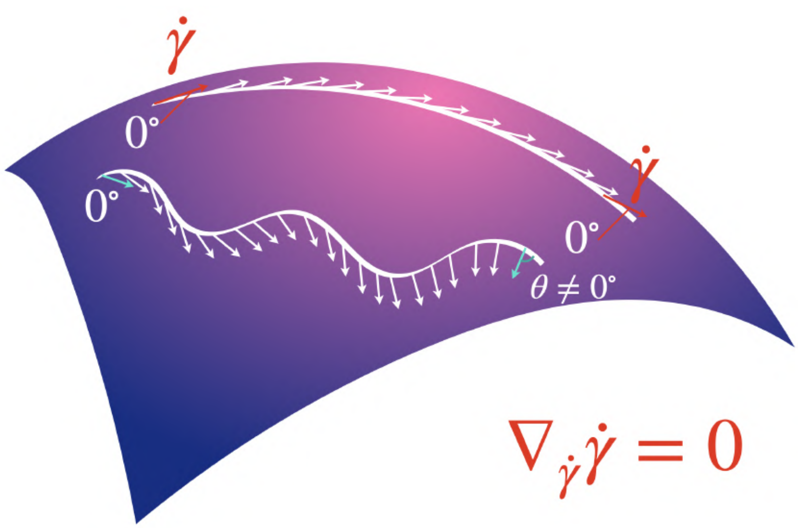
A Killing vector is a direction you can move in without changing distances in the space: no stretching or contracting along it. We won't get into details here, but more rigorously speaking, a Killing vector is a vector field whose flow preserves the metric. The distances and angles remain unchanged along it.





Note: *If you'd like to be the first to find out when we launch our very first books and courses, sign up with your email address on our homepage, dibeos.net.*

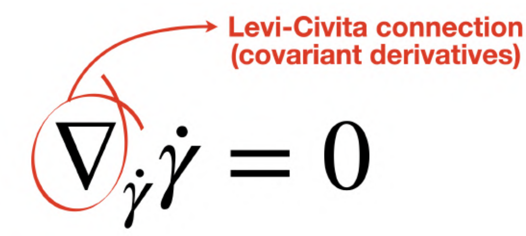
Before we dive deeper into this first method, though, let's quickly recall what a *geodesic equation* actually is.



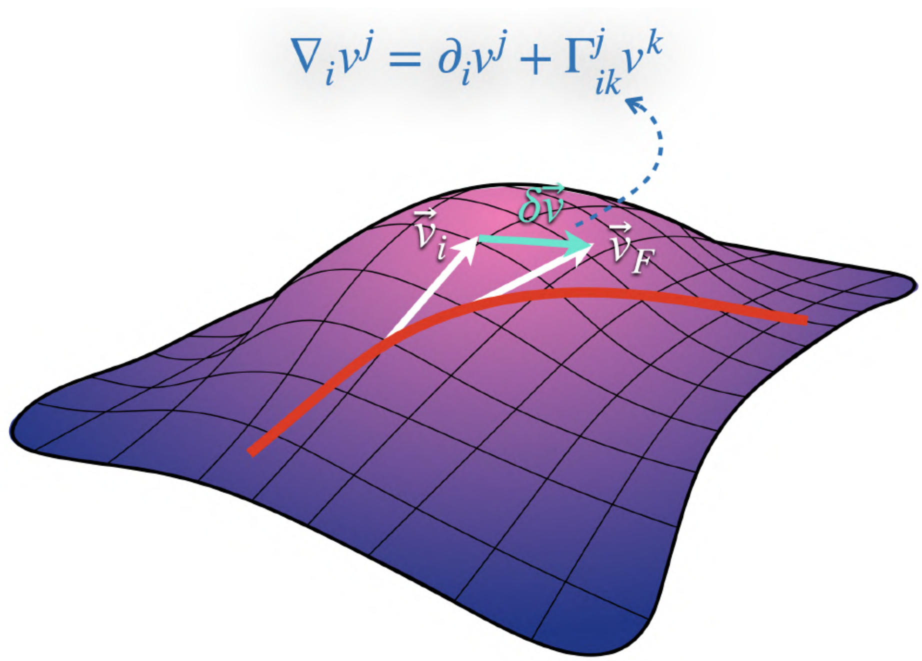
In **Differential Geometry**, a geodesic is a curve whose velocity vectors (i.e. its tangent vectors at each point) stay parallel to itself as you move along it. In more formal terms, it's a curve $\gamma(t)$ whose tangent vector $\dot{\gamma}(t)$ satisfies this relation:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0$$

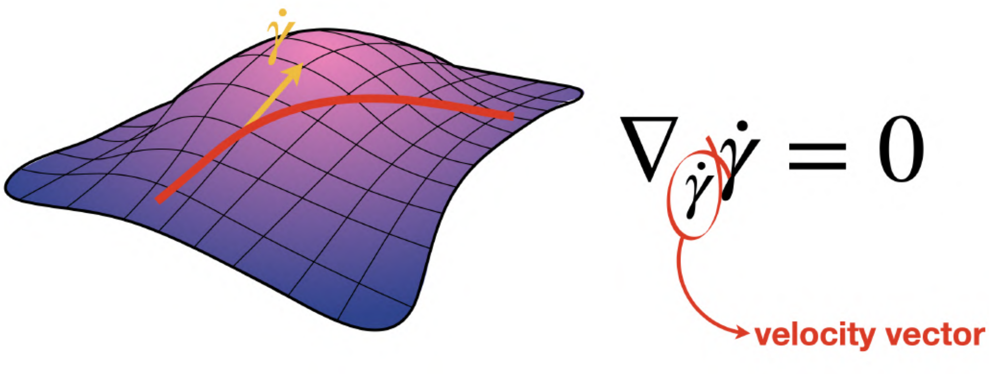
With this symbol ∇ representing the *Levi-Civita connection*, which is just one of the many types of *covariant derivatives* out there.



A covariant derivative measures the infinitesimal rate of change of a vector field, taking into account the manifold's curvature.



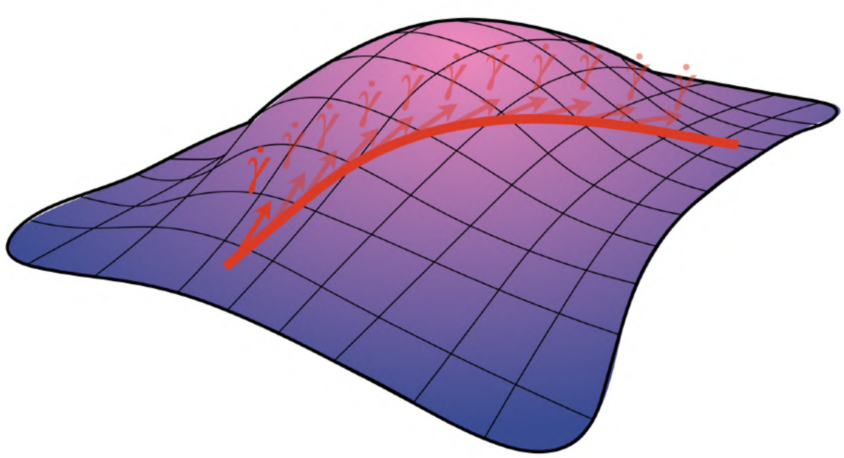
The subscript in $\nabla_{\dot{\gamma}}$ tells us to measure that change in the direction of the curve's own velocity vector.



This equation is basically telling us:

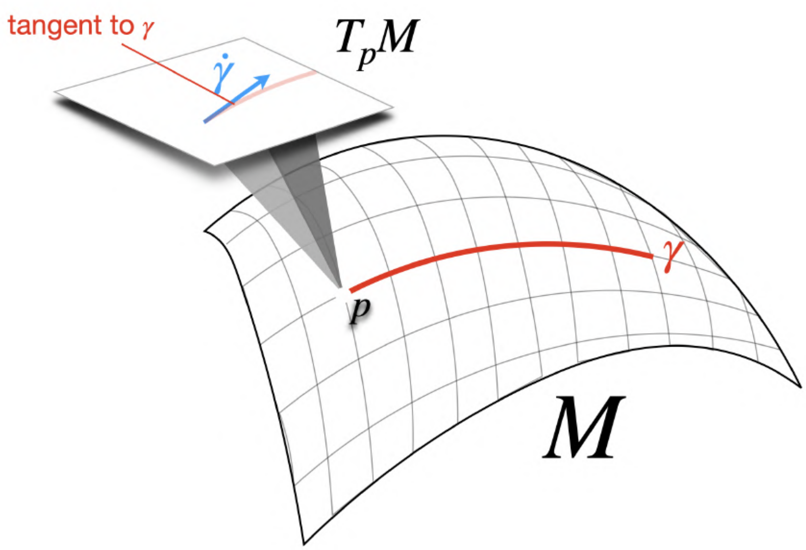
"The tangent vector to the curve has <u>zero</u> change in its own direction. It's parallel transported along the curve, i.e. no change at all!"

The vector stays parallel to itself along the entire curve γ.

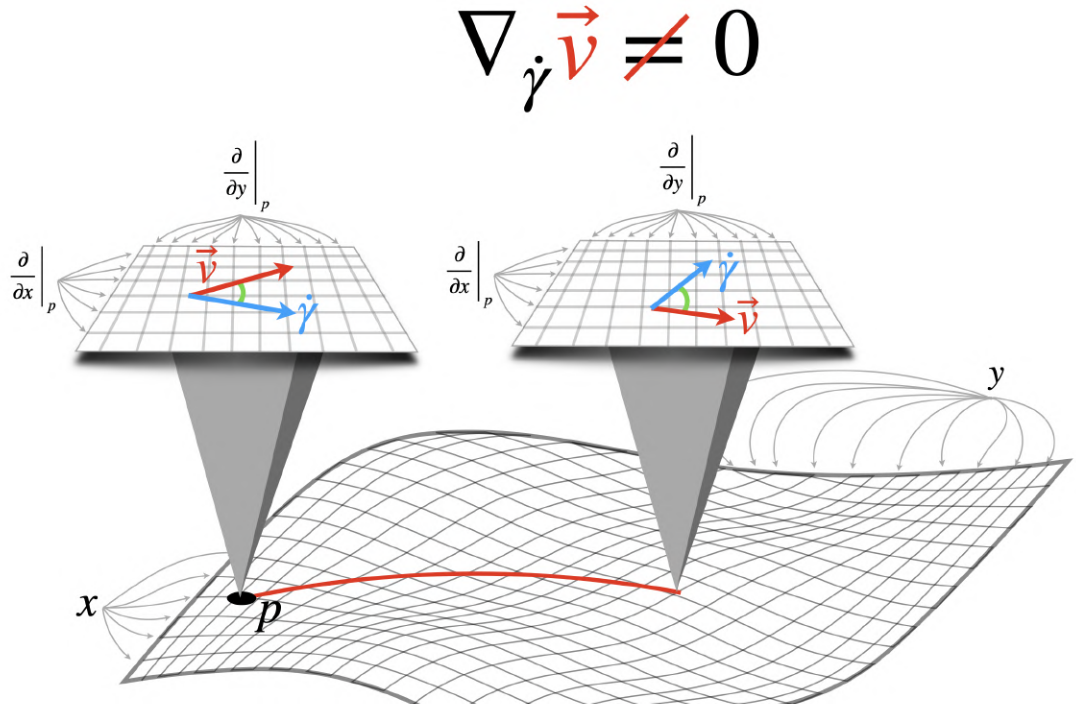


It's important to mention one thing that got me really confused when I first learned these concepts years ago:

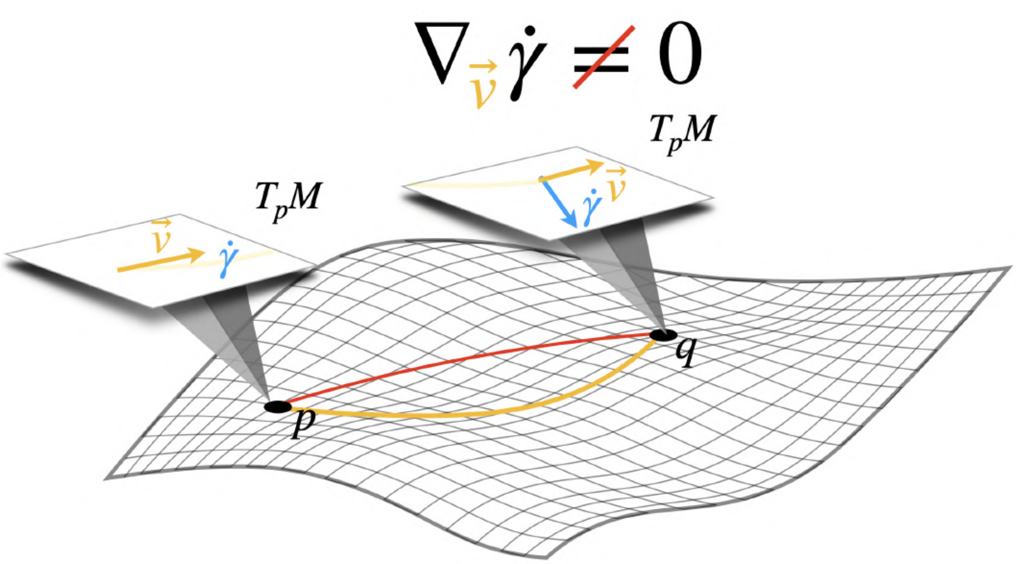
γ is the path, or curve, in the manifold M. $\dot{\gamma}$ is the velocity vector at a point $p \in M$, i.e. the vector that is not only tangent to the manifold M, and therefore lives in the tangent space T_pM, but it's also tangent to the curve γ itself.



This is different from a random vector $\vec{v} \in T_pM$, and as a consequence the "magic" of getting zero in the RHS ($\equiv$ right-hand side) is not guaranteed anymore, even if you are transporting it along a geodesic path γ:



The opposite is also true, i.e. if you consider the velocity vector of a geodesic curve γ (so, the vector that is tangent to the manifold AND to the geodesic γ itself), and parallel transport it along any other random direction given by a vector $\vec{v}$, from point p to point q, the vector does not keep its original direction necessarily:



The vector $\dot{\gamma}$, then, can be written as a linear combination of the basis vectors of the space where it lives. And since it lives in the tangent space, the basis is $\left\{\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}\right\}$, or simply: $\left\{\frac{\partial}{\partial x^i}\right\}$, where $i \in \{1, \ldots, n = \dim(M)\}$.

$$\dot{\gamma} = \boxed{?}^i \frac{\partial}{\partial x^i} = \boxed{?}^1 \frac{\partial}{\partial x^1} + \boxed{?}^2 \frac{\partial}{\partial x^2} + \cdots + \boxed{?}^n \frac{\partial}{\partial x^n}$$

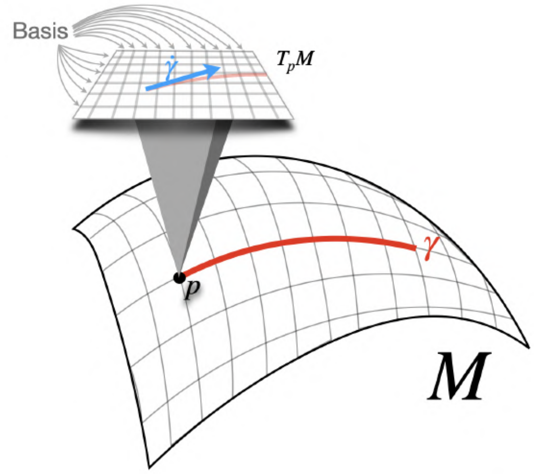
Suggestion: *If you're finding these concepts confusing, check out the following video on our channel and the detailed PDF, where we explain* ***Differential Forms*** *from the ground up:*

The Core of Differential Forms

PDF link: Differential Forms

$$\left\{ \frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n} \right\} = \left\{ \frac{\partial}{\partial x^i} \right\}$$

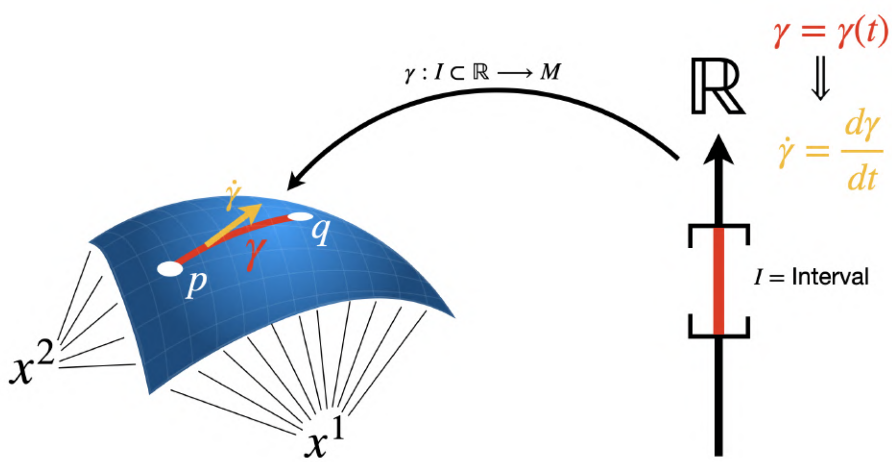
$$i \in \{1, \ldots, n = \dim(M)\}$$



$$\dot{\gamma} = \boxed{?}^i \frac{\partial}{\partial x^i} = \boxed{?}^1 \frac{\partial}{\partial x^1} + \boxed{?}^2 \frac{\partial}{\partial x^2} + \cdots + \boxed{?}^n \frac{\partial}{\partial x^n}$$

Now, we need to figure out what the components of this linear combination are... $\boxed{?}^i$.

Well, let's go back to first principles: what is the curve γ? It's just an infinite set of points in the manifold that when put together form a smooth continuous path that we can parametrize with $t \in \mathbb{R}$.



$$\gamma = \left(x^1(t), x^2(t), \ldots, x^n(t)\right) = \left(x^i(t)\right) \qquad i \in \{1, \ldots, n\}$$

$\dot{\gamma}$ is just a notation to its differential with respect to the parameter t (or time, if you will):

$$\dot{\gamma} = \frac{d\gamma}{dt}$$

And since γ is just an infinite set of points in the manifold, we can write it in local coordinates as this:

$$\begin{aligned} \gamma &= \left(x^1(t),\, x^2(t),\, \ldots,\, x^n(t)\right) = \left(x^i(t)\right) = \\ &= x^i(t)\,\frac{\partial}{\partial x^i} \qquad\qquad i \in \{1, \ldots, n\} \end{aligned}$$

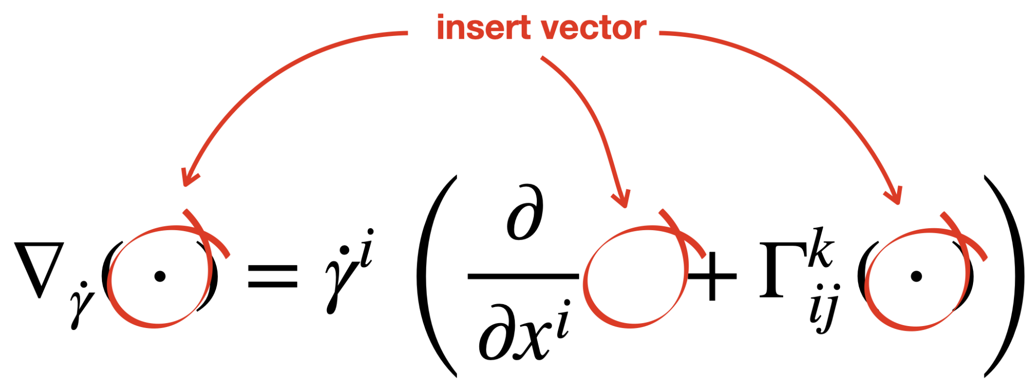
And therefore:

$$\dot{\gamma} = \frac{d\gamma}{dt} = \frac{d}{dt}\left(x^i(t)\right)\,\frac{\partial}{\partial x^i} = \dot{x}^i\,\frac{\partial}{\partial x^i}$$

$$\therefore \boxed{\dot{\gamma} = \dot{x}^i\,\frac{\partial}{\partial x^i}}$$

Great! Just to remind you guys: our goal is to use this definition of velocity vector $\dot{\gamma}$, of the geodesic path γ, in the equation $\nabla_{\dot{\gamma}}\dot{\gamma} = 0$.

All we need to do now is to find an explicit way of writing the Levi-Civita connection ∇.



The Levi-Civita connection can be seen as a mapping (or operator) that acts on a vector (that's where we will insert this vector – in this red slot below):

$$\nabla_{\dot{\gamma}}(\,\cdot\,) = \dot{\gamma}^i \left(\frac{\partial}{\partial x^i} \quad + \Gamma^k_{ij}\,(\,\cdot\,) \right)$$

It's important to say that the way we wrote it here is not standard, and you'll probably not find it written this way anywhere else, but it does help us to understand how this operator acts on vectors.

The vector is written as $\vec{v} = v^k \frac{\partial}{\partial x^k}$.

$$\nabla_{\dot{\gamma}}\,(\,\vec{v}\,) = \dot{\gamma}^i \left(\frac{\partial}{\partial x^i} \quad + \Gamma^k_{ij}\,(\,\cdot\,) \right)$$

This partial term (in green below) differentiates the components v^k of the vector field:

$$\nabla_{\dot{\gamma}}(\vec{v}) = \dot{\gamma}^i \left(\frac{\partial}{\partial x^i} \quad + \Gamma^k_{ij} (\,\cdot\,) \right)$$

Now, this term (below) means "**move the vector $\vec{v}$ in the direction of the path's velocity**".

"move the vector $\vec{v}$ in the direction of the path's velocity"

$$\nabla_{\dot{\gamma}}(\vec{v}) = \boxed{\dot{\gamma}^i \left(\frac{\partial}{\partial x^i} \, v^k \right.} + \Gamma^k_{ij} (\,\cdot\,) \Big)$$

And finally, the Christoffel symbols correct for the fact that the basis vectors themselves change as you move, because of the manifold's curvature.

$$\nabla_{\dot{\gamma}}\vec{v} = \dot{\gamma}^i \left(\frac{\partial v^k}{\partial x^i} + \Gamma^k_{ij} \, v^j \right) \frac{\partial}{\partial x^k}$$

Since we want the vector $\vec{v}$ to match the velocity vector $\dot{\gamma} = \dot{x}^i \frac{\partial}{\partial \dot{x}^i}$ of the geodesic path γ, we substitute the vector $\vec{v}$ with $\dot{\gamma}$:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = \dot{\gamma}^i \left(\frac{\partial \dot{x}^k}{\partial x^i} + \Gamma^k_{ij} \, \dot{x}^j \right) \frac{\partial}{\partial x^k} \implies$$

But what is $\dot{\gamma}^i$ here? This is the i-th component of the vector $\dot{\gamma} = \dot{x}^i \frac{\partial}{\partial x^i}$. I.e., just $\dot{x}^i$.

$$\implies \quad \nabla_{\dot{\gamma}} \dot{\gamma} = \dot{x}^i \left(\frac{\partial \dot{x}^k}{\partial x^i} + \Gamma^k_{ij}\, \dot{x}^j \right) \frac{\partial}{\partial x^k} = 0$$

And if all of it is zero, according to our geodesic condition, then the following term (which is just a component of the vector) is zero:

$$\nabla_{\dot{\gamma}} \dot{\gamma} = \boxed{\dot{x}^i \left(\frac{\partial \dot{x}^k}{\partial x^i} + \Gamma^k_{ij}\, \dot{x}^j \right)} \frac{\partial}{\partial x^k} = 0 \quad \implies \quad \dot{x}^i \left(\frac{\partial \dot{x}^k}{\partial x^i} + \Gamma^k_{ij}\, \dot{x}^j \right) = 0 \quad \implies$$

$$\left(\underbrace{\frac{d\dot{x}^i}{dt} \cdot \frac{\partial \dot{x}^k}{\partial x^i}}_{=1} \right) + \Gamma^k_{ij}\, \dot{x}^i\, \dot{x}^j = 0 \quad \implies$$

$$\implies \quad \boxed{\frac{d^2 x^k}{dt^2} + \Gamma^k_{ij}\, \dot{x}^i\, \dot{x}^j = 0}$$

And that's the famous geodesic equation!

Where $i, j, k \in \{1, \ldots, n = \dim(M)\}$.

And that's precisely the equation we want to solve.

Back to our first method now: (**1 Symmetries & Killing Vectors**)

Just to recap, we will use mathematical objects called Killing vectors in this method in order to explore symmetries of the manifold and thus solve the geodesic equation.

A Killing vector is a vector field whose flow preserves the metric. No stretching or contracting along it, even though the manifold itself does so (due to curvature).

Just to help you guys to better grasp the abstract notions we're about to deal with, let's see what these Killing vector fields could represent in physics, for example (more specifically, in the spacetime manifold):

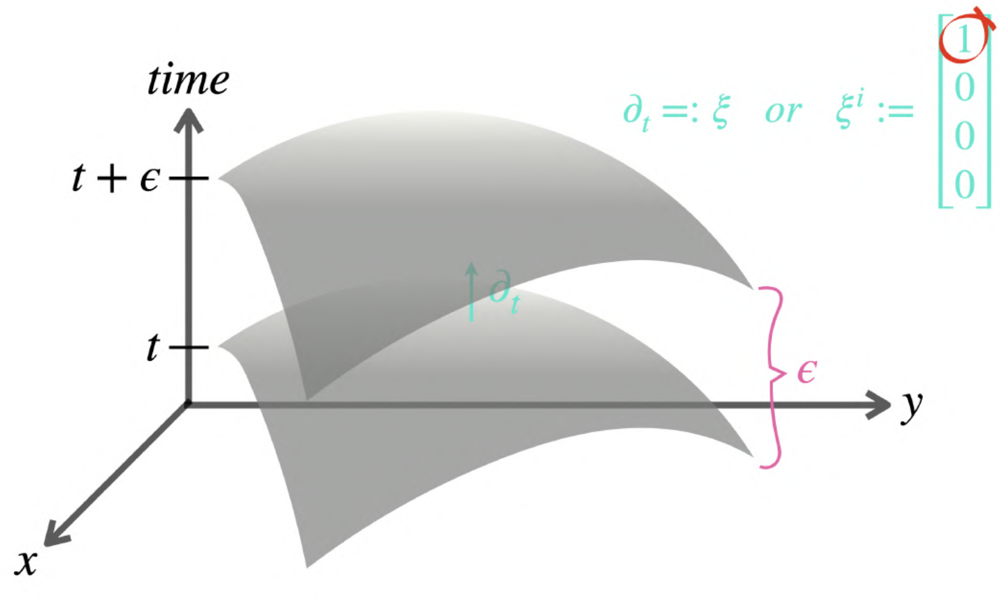
(1) Time translation symmetry: (energy conservation)

The Killing field is $\frac{\partial}{\partial t}$, or simply ∂_t.

If the metric (spacetime geometry) does not depend on time t, then time translations $t \longrightarrow t + \epsilon$ are a type of symmetry. The Killing vector field ∂_t points along the time direction, and it can also be expressed as

$$\partial_t =: \xi \quad or \quad \xi^i := \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

(The only non-zero component is in the time direction – first slot)



Now, take this Killing vector, lower its index using the metric g_{ij}, and then multiply it (using the inner product) to the velocity vector $\dot{x}^j$:

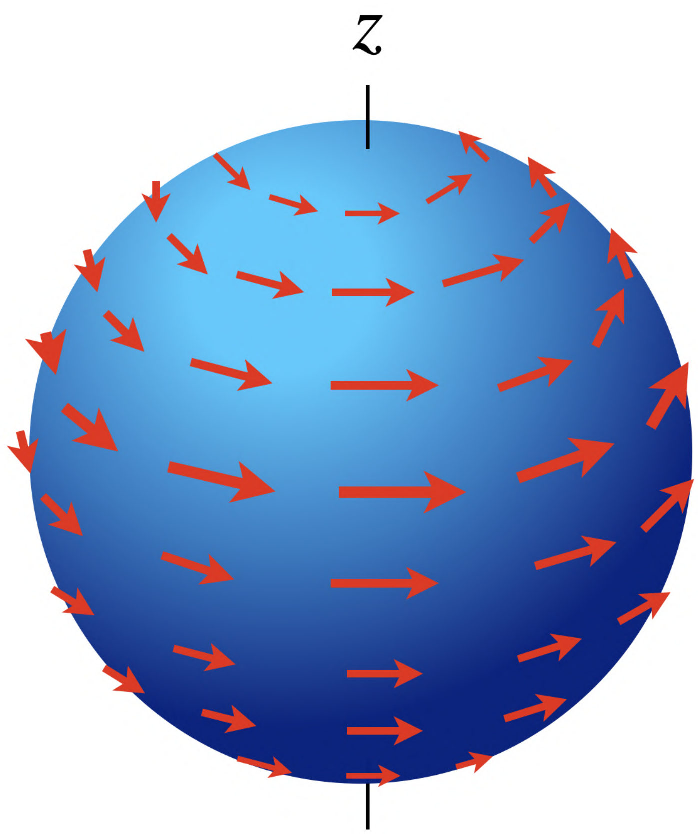
$$K = g_{ij}\,\xi^i\,\dot{x}^j$$

This quantity (K) is a scalar (a number) that is the projection (or "influence") that the Killing vector has on the direction of the velocity vector in local coordinates. K is conserved. In many contexts, including in the **Schwarzschild**, **Minkowski** and **FRW** spacetime solutions, K represents the *energy per unit mass* of the particle moving along the geodesic.

Anyway, there are other Killing vectors as well:

(2) Spatial rotation symmetry: (angular momentum conservation)

Suppose that the geometry is invariant under rotations around some axis (say the z-axis).



Then, the corresponding Killing field is the generator of that rotation:

$$\xi = \begin{bmatrix} -y \\ x \end{bmatrix} = -y\,\partial_x + x\,\partial_y$$

This Killing vector expresses an infinitesimal rotation in the xy plane.

Plugging it into the same equation as before $(K = g_{ij}\,\xi^i\,\dot{x}^j)$ gives us a new conserved quantity, namely the *angular momentum* component around the z-axis.

(3) Spatial translation symmetry: (linear momentum conservation)

If the metric is invariant under shifts in the x-direction (like in flat Minkowski spacetime), then $\xi := \partial_x$ is a Killing vector.

The conserved quantity from $K = g_{ij}\,\xi^i\,\dot{x}^j$, this time, is the x-component of *linear momentum*. And so on and so forth for the other components:

$$\boxed{K_x = p_x} \qquad \boxed{K_y = p_y} \qquad \boxed{K_z = p_z}$$

$$\vec{p} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

We will not get into detail here, but let's suppose we're studying the 2-sphere manifold $\left(\mathbb{S}^2\right)$. In order to find geodesic paths on the sphere, i.e. to find solutions of the geodesic equations $\boxed{\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j = 0}$, we first need to define a metric. The standard one here is in spherical coordinates:

$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

or

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix} \qquad i, j \in \{\theta\,,\,\phi\}$$

Once we have a metric, i.e. a rule for measuring distances and angles, we can calculate the Christoffel symbols:

$$\Gamma^{\theta}_{\phi\phi} = -\sin\theta\,\cos\theta \qquad \Gamma^{\phi}_{\theta\phi} = \Gamma^{\phi}_{\phi\theta} = \cot\theta$$

(All the others are zero)

Plugging these results into the geodesic equation $\boxed{\ddot{x}^k + \Gamma^k_{ij}\,\dot{x}^i\,\dot{x}^j = 0}$, for $i, j \in \{\theta\,,\,\phi\}$, we get these two **coupled nonlinear second-order ODEs**:

$$\begin{cases} \ddot{\theta} - \sin\theta\,\cos\theta\,\dot{\phi}^2 = 0 & (I) \\ \ddot{\phi} + 2\cot\theta\,\dot{\theta}\,\dot{\phi} = 0 & (II) \end{cases}$$

They are hard to solve! But if you can find solutions, then these paths γ will be geodesic curves, which in this particular case would draw great circles around the sphere.

Now, what happens if we approach the problem with Killing vectors instead?!

Well, the situation clearly has rotational symmetry. So, why not taking advantage of that? The Killing vector is $\xi = \partial_\phi$.

Thus, the conserved quantity K here is:

$$K = g_{ij}\,\xi^i\,\dot{x}^j = g_{\theta\theta}\,\xi^\theta\,\dot{x}^\theta + g_{\theta\phi}\,\xi^\theta\,\dot{x}^\phi + g_{\phi\theta}\,\xi^\phi\,\dot{x}^\theta + g_{\phi\phi}\,\xi^\phi\,\dot{x}^\phi = \sin^2\theta\,\dot{\phi} \quad \Longrightarrow$$

$$\Longrightarrow \quad \boxed{K = \sin^2\theta\,\dot{\phi}} \quad \text{is the } \textbf{angular momentum}.$$

We can then write:

$$\boxed{\dot{\phi} = \frac{K}{\sin^2\theta}} \qquad (*)$$

And finally, substituting $(*)$ into equations (I) and (II) we get:

$$\begin{cases} \ddot{\theta} = \dfrac{K^2\cos\theta}{\sin^3\theta} \\ \dot{\phi} = \dfrac{K}{\sin^2\theta} \end{cases}$$

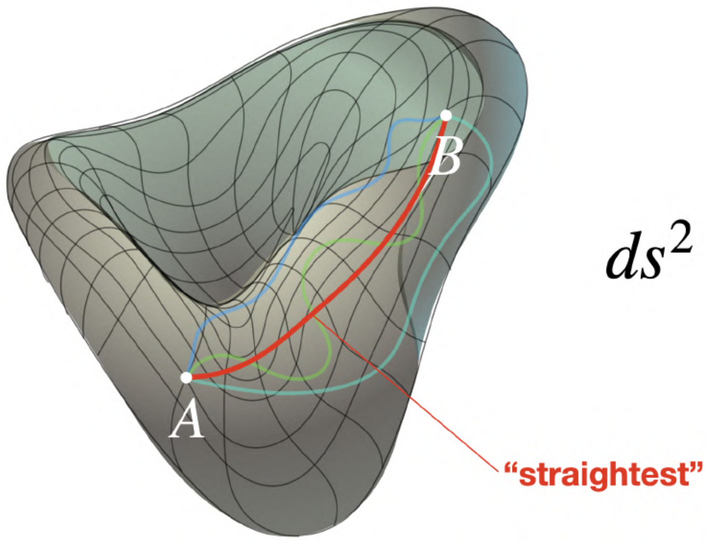
I know, that looking at them doesn't seem so, but they are way more manageable to look for solutions! Instead of solving two **coupled second-order ODEs**, we reduced to one **second-order ODE in** θ, and a **simple first-order relation for** ϕ.

All of it thanks to *Killing vectors*.

2 First Integral / Conservation Laws

Imagine we have a manifold with some metric defined on it.

Out of all possible paths connecting points A and B in a manifold, there is one that is the "straightest" possible curve.

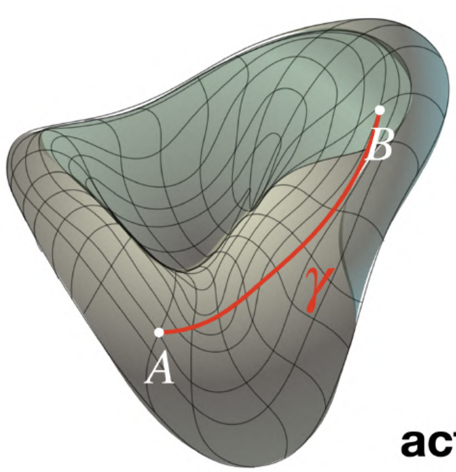


$$ds^2 = g_{ij}\, dx^i\, dx^j$$

Using the **Principle of Least Action** (from Variational Calculus), we can derive the Lagrangian of the system, which will allow us to find the geodesic path we want:

$$\mathcal{L} = \frac{1}{2} g_{ij}\, \dot{x}^i\, \dot{x}^j$$

Think of the Lagrangian as a measure of the ”cost” of a path.



Lagrangian:

$$\mathscr{L} = \frac{1}{2} g_{ij}\, \dot{x}^i\, \dot{x}^j$$

geodesic

action: $\mathcal{S}[\gamma] = \int_\gamma \mathscr{L}\, dt$

And the geodesic is what you get once you minimize the total cost, which is the action of the system, and is built from the Lagrangian:

$$\mathcal{S}[\gamma] = \int_\gamma \mathcal{L}\, dt$$

There are two natural sources of *conserved quantities*:

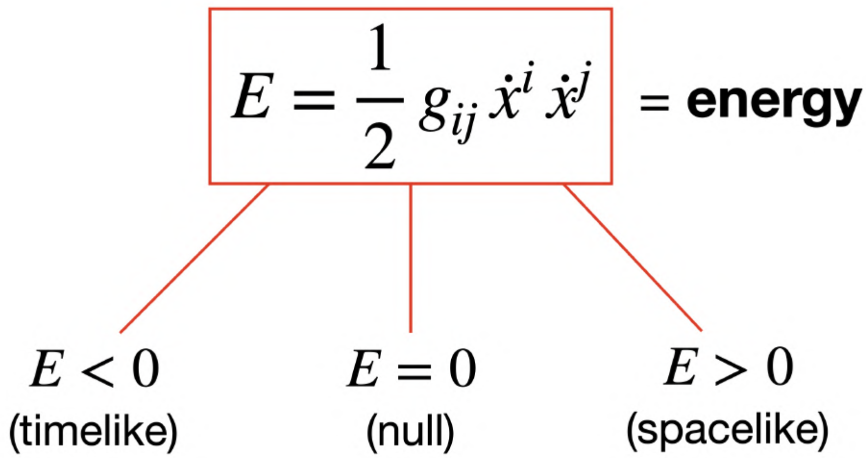
(1) Energy-like quantity:

When the Lagrangian does not depend explicitly on the parameter t (which is often interpreted as proper time in physics), then there is some quantity here that's being conserved. This result is called **Noether's Theorem**.

The conserved quantity here is identical to the Lagrangian, but is interpreted differently:

$$E = \frac{1}{2} g_{ij}\, \dot{x}^i\, \dot{x}^j$$

When studying mechanics, this is the usual physical energy of the system. But in purely mathematical terms, it is a constant that classifies geodesics as *timelike* $(E < 0)$, *null* $(E = 0)$ or *spacelike* $(E < 0)$.



The fact that the Lagrangian does not depend explicitly on the parameter t means that there is symmetry under time translations. And by Noether's theorem, this symmetry guarantees that this quantity (E) is conserved.

Ok, but from the intuitive point of view, what do these terms mean?

Timelike *Null* *Spacelike*

There 3 types of geodesics:

(I) Timelike: $(E < 0)$ These are trajectories of massive particles, like protons, neutrons, electrons and so on.

(II) Null: $(E = 0)$ These are trajectories of massless particles that move at the speed of light, like photons and gluons.

(III) Spacelike: $(E > 0)$ These are "faster-than-light" trajectories, which are not physically possible, but in the context of Pure Math they are totally valid.

Let's see the second natural source of conserved quantity, before moving on to the third method:

(2) Cyclic Coordinates: (or Generalized Momenta)

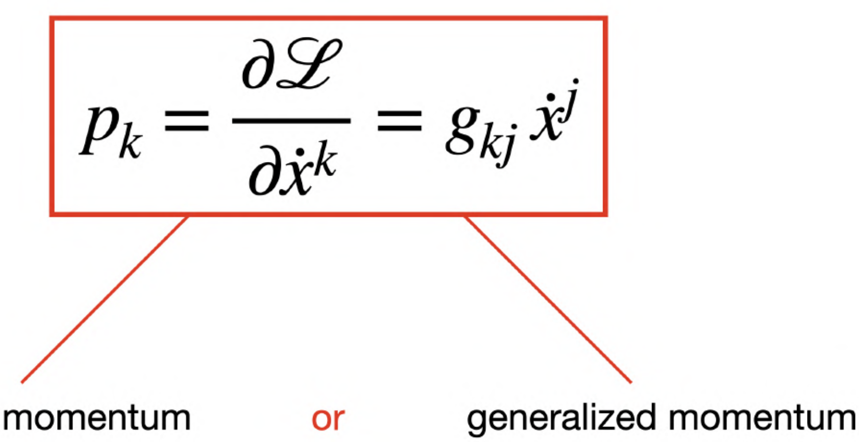
When the Lagrangian does not depend on a specific coordinate x^k, just like in our case here:

$$\mathcal{L} = \frac{1}{2} g_{ij} \, \dot{x}^i \, \dot{x}^j$$

, then this quantity is conserved:

$$p_k = \frac{\partial \mathcal{L}}{\partial \dot{x}^k} = g_{kj} \, \dot{x}^j$$

It is called the **conjugate momentum**, or **generalized momentum**.



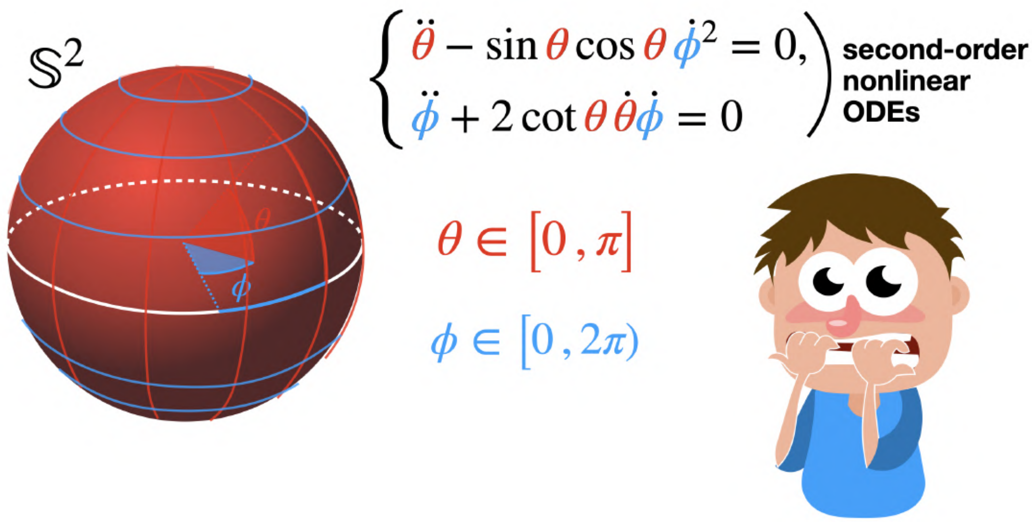
Every coordinate x^1 , x^2 , ... , x^n has an associated conjugate momentum.

From the physics point of view: If a coordinate x^k does not appear explicitly in the Lagrangian, like in our case for all coordinates $\left(\mathcal{L} = \frac{1}{2} g_{ij} \, \dot{x}^i \, \dot{x}^j\right)$, then moving along any coordinate direction doesn't change the "cost". This symmetry implies another conservation law (once again because of Noether's theorem).

For example, since the coordinate x does not appear in $\mathcal{L}$, then $p_x = \frac{\partial \mathcal{L}}{\partial \dot{x}}$ is conserved, i.e. we have *linear momentum conservation*.

For a coordinate angle ϕ that doesn't appear in $\mathcal{L}$, we have $p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}}$, i.e. *angular momentum conservation*.

Ok, but how does it help us to solve the geodesic equations? Well, it simplifies them a lot!



With all of it in place, let's see how this formalism allows us to simplify the geodesic equation, and then find solutions, in the case of the sphere $\mathbb{S}^2$, for example.

The metric is:

$$g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

The Lagrangian:

$$\mathcal{L} = \frac{1}{2} g_{ij}\, \dot{x}^i\, \dot{x}^j = \frac{1}{2} \left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right)$$

Energy conservation:

$$E = \frac{1}{2} \left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right) = \text{constant}$$

Angular momentum conservation:

$$p_\phi = \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \frac{\partial}{\partial \dot{\phi}} \left[\frac{1}{2} \left(\dot{\theta}^2 + \sin^2\theta\, \dot{\phi}^2 \right) \right] =$$

$$= \frac{2}{2} \sin^2\theta\dot{\phi} \implies \sin^2\theta\, \dot{\phi} = \text{constant}$$

Let's call this constant L (angular momentum):

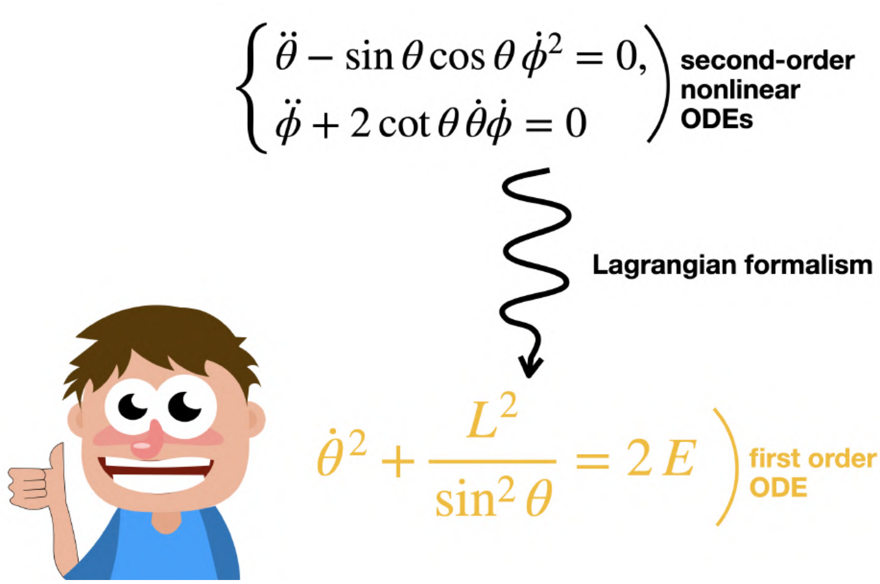
$$\sin^2\theta\,\dot{\phi} =: L \quad\Longrightarrow\quad \boxed{\dot{\phi} = \frac{L}{\sin^2\theta}}$$

Now, substituting this expression of $\dot{\phi}$ into the energy equation, gives us:

$$E = \frac{1}{2}\left(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2\right) = \frac{1}{2}\dot{\theta} + \frac{1}{2}\cdot\frac{L^2}{\sin^2\theta} \quad\Longrightarrow$$

$$\Longrightarrow \quad \boxed{\dot{\theta}^2 + \frac{L^2}{\sin^2\theta} = 2E}$$

, which is a **first-order ODE in a single variable** $\theta(t)$. It eliminates the coupling with $\phi(t)$ and simplifies the problem a lot!



Note: *Why is this method called the "First Integral" if there are no actual integrals involved?*

In the context of Differential Equations, a "first integral" is any quantity that remains constant along the solutions of that equation. In other words, it's a function $I\left(x, \dot{x}\right)$ *such that*

$$\frac{d}{dt} I\left(x(t), \dot{x}(t)\right) = 0$$

along every solution $x(t)$.

3 Separation of Variables

This is probably the physicists' favorite method to solve differential equations, because of its simplicity. Unfortunately however, it rarely works since it requires some very special conditions, such as the possibility to rewrite the PDE in a way that each side (i.e. LHS and RHS) depends only on one variable.

If it's not possible, you might need some more sophisticated techniques, like *Fourier analysis*, *Green's functions* or *numerical methods*.

Anyway, before applying it to the geodesic problem, let's see a simple example in order to understand the gist of this method:

$$\nabla^2 u = 0$$

This is the famous **Laplace equation** used in Pure and Applied Math, such as in the calculus of the electrostatic potential in a region with no charges, for instance.

Considering the problem in $2D$, this equation can be expanded into:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

So let's assume (and that's a strong assumption, by the way) that there is a solution of the form $u(x,y) = X(x)\,Y(y)$. In other words, the solution can be written as the multiplication of 2 separate functions: one of them, $X(x)$, depending only on x, and the other one, $Y(y)$, depending only on y.

Let's plug this solution into our Laplace equation:

$$\frac{\partial^2}{\partial x^2}\left(X(x)\,Y(y)\right) + \frac{\partial^2}{\partial y^2}\left(X(x)\,Y(y)\right) = 0 \implies$$

$$\implies \frac{d^2X}{dx^2}(x)\,Y(y) + X(x)\,\frac{d^2Y}{dy^2}(y) = 0 \implies$$

$$\implies \frac{d^2X}{dx^2}(x)\,Y(y) = -X(x)\,\frac{d^2Y}{dy^2}(y)$$

Now, we divide everything by $X(x)\,Y(y)$ (assuming $X(x) \neq 0$ and $Y(y) \neq 0$):

$$\frac{1}{X(x)\,\cancel{Y(y)}} \cdot \frac{d^2X}{dx^2}(x)\,\cancel{Y(y)} = \frac{1}{\cancel{X(x)}\,Y(y)} \cdot \left(-\cancel{X(x)}\right)\frac{d^2Y}{dy^2}(y) \implies$$

$$\implies \boxed{\frac{1}{X(x)} \cdot \frac{d^2X}{dx^2}(x) = -\frac{1}{Y(y)} \cdot \frac{d^2Y}{dy^2}(y)}$$

Notice how the LHS depends **only** on x and the RHS depends **only** on y, and yet they are the same (equal sign between them)! There is only one way this could be true: if each side equals a constant (λ).

$$\boxed{\frac{1}{X(x)} \cdot \frac{d^2X}{dx^2}(x) =: \lambda} \qquad \boxed{\frac{1}{Y(y)} \cdot \frac{d^2Y}{dy^2}(y) =: -\lambda}$$

λ is called: the *separation constant*.

At this point we reduced the problem to 2 very simple differential equations to solve.

For $\lambda > 0$:

$$(I) \quad \boxed{\frac{1}{X(x)} \cdot \frac{d^2X}{dx^2}(x) =: \lambda} \quad \implies \quad \frac{d^2X}{dx^2}(x) - \lambda\, X(x) = 0$$

Solutions are *exponential*:

$$\boxed{X(x) = C\, e^{\sqrt{\lambda}\, x} + D\, e^{-\sqrt{\lambda}\, x}}\,, \qquad C, D \text{ are constants.}$$

$$(II) \quad \boxed{\frac{1}{Y(y)} \cdot \frac{d^2Y}{dy^2}(y) =: -\lambda} \quad \implies \quad \frac{d^2Y}{dy^2}(y) + \lambda\, Y(y) = 0$$

Solutions are *oscillatory* (sine and cosine):

$$\boxed{Y(y) = A\,\cos\left(\sqrt{\lambda}\,y\right) + B\,\sin\left(\sqrt{\lambda}\,y\right)}\ , \qquad A, B \text{ are constants.}$$

These constants A, B, C, D are determined according to the **initial conditions** (also called **Cauchy conditions**) or to the **boundary conditions** (which can be **Dirichlet**, **Neumann**, or **Robin conditions**) of the specific problem you're trying to solve.

For $\lambda = 0$:

$$(I) \quad \boxed{\frac{1}{X(x)} \cdot \frac{d^2X}{dx^2}(x) =: 0} \quad \Longrightarrow \quad \frac{d^2X}{dx^2}(x) = 0 \quad \Longrightarrow \quad \boxed{X(x) = A\,x + B}$$

$$(II) \quad \boxed{\frac{1}{Y(y)} \cdot \frac{d^2Y}{dy^2}(y) =: 0} \quad \Longrightarrow \quad \frac{d^2Y}{dy^2}(y) = 0 \quad \Longrightarrow \quad \boxed{Y(y) = C\,y + D}$$

For $\lambda < 0$: (say $\lambda = -\mu\ ;\ \mu > 0$)

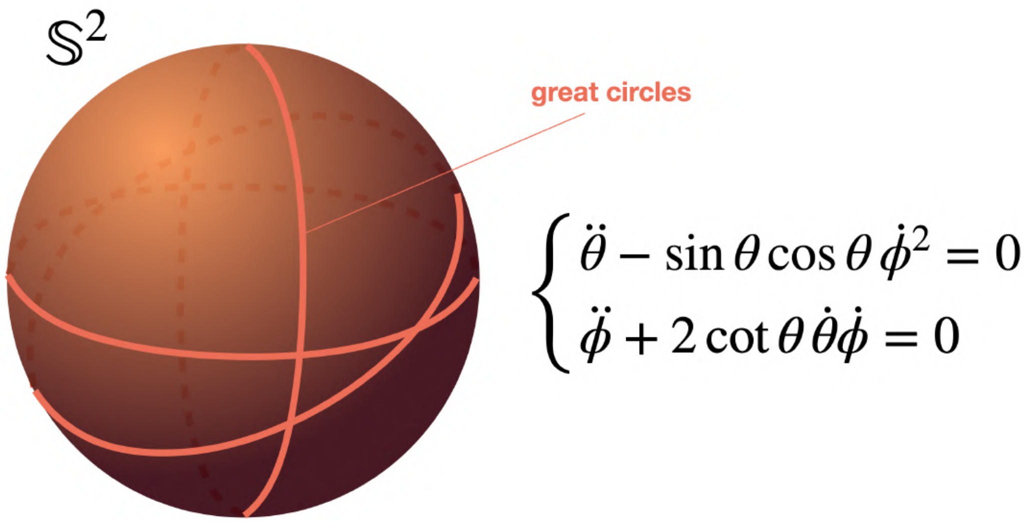
$$(I) \quad \boxed{\frac{1}{X(x)} \cdot \frac{d^2X}{dx^2}(x) =: -\mu} \quad \Longrightarrow \quad \boxed{X(x) = A\,\cos\left(\sqrt{\mu}\,x\right) + B\,\sin\left(\sqrt{\mu}\,x\right)}$$

$$(II) \quad \boxed{\frac{1}{Y(y)} \cdot \frac{d^2Y}{dy^2}(y) =: \mu} \quad \Longrightarrow \quad \boxed{Y(y) = C\, e^{\sqrt{\mu}\, y} + D\, e^{-\sqrt{\mu}\, y}}$$

Going back to our geodesic equations in the sphere, we had:

$$\begin{cases} \ddot{\theta} - \sin\theta\, \cos\theta\, \dot{\phi}^2 = 0 \\ \ddot{\phi} + 2\cot\theta\, \dot{\theta}\, \dot{\phi} = 0 \end{cases}$$

Just as a reminder, geodesics in this manifold are just great circles.



Let's try to separate variables. Consider the second equation and rewrite it as

$$\ddot{\phi} = -2\,\cot(\theta)\dot{\theta}\dot{\phi}$$

Define:

$$u := \dot{\phi} \implies \ddot{\phi} = \frac{du}{dt}$$

Then:

$$\frac{du}{dt} = -2\ \cot(\theta)\dot{\theta}u$$

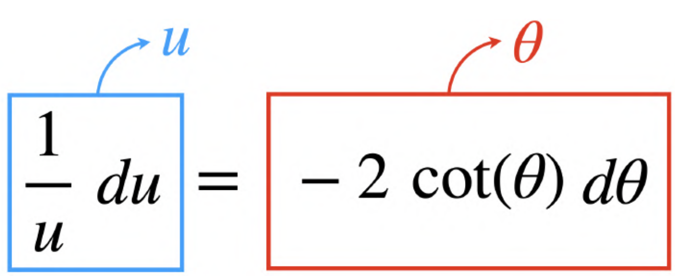
Using the chain rule we can rewrite $\frac{du}{dt} = \frac{du}{d\theta}\dot{\theta}$:

$$\frac{du}{dt}\dot{\theta} = -2\ \cot(\theta)\dot{\theta}u$$

Divide both sides by $\dot{\theta}$:

$$\frac{1}{u} \cdot \frac{du}{dt} = -2\ \cot(\theta)$$

Notice how the LHS depends explicitly only on u, and the RHS depends explicitly only on θ.



We can integrate both sides:

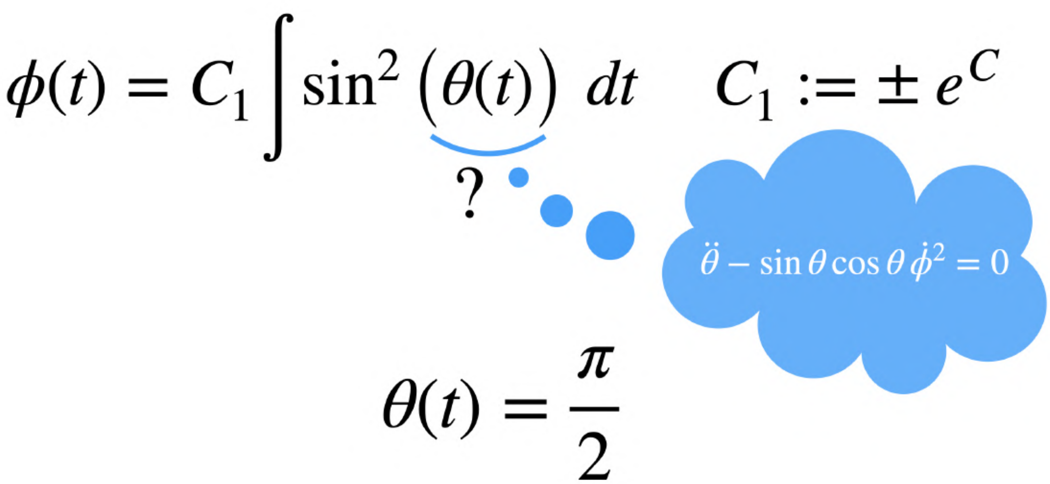
$$\int \frac{1}{u}\, du = \int -2\, \cot(\theta)\, d\theta \quad \Longrightarrow$$

$$\Longrightarrow \quad \ln|u| = 2 \ln|\sin(\theta)| + C \quad \Longrightarrow$$

$$\Longrightarrow \quad |u| = C' sin^2(\theta) \quad \Longrightarrow \quad \dot{\phi} = C_1 \sin^2(\theta) \quad \Longrightarrow$$

$$\therefore \; \boxed{\phi(t) = C_1 \int \sin^2(\theta(t))\, dt} \quad C_1 := \pm e^C$$

However, we still need to know $\theta(t)$ from the first geodesic equation in order to calculate this integral explicitly.

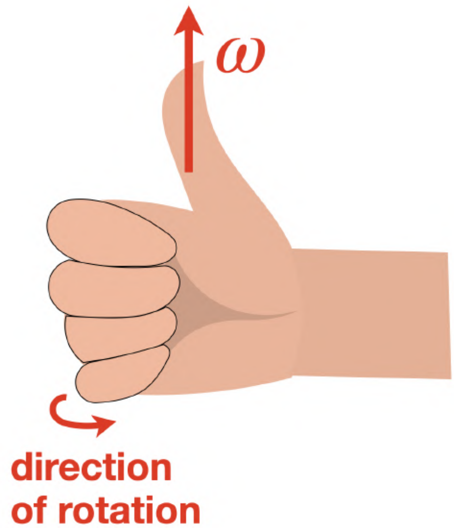


For example, pick $\theta(t) := \frac{\pi}{2}$.

$$\phi(t) = C_1 \int \sin^2\left(\frac{\pi}{2}\right)\, dt = C_1 \int 1 \cdot dt = C_1 \cdot t$$

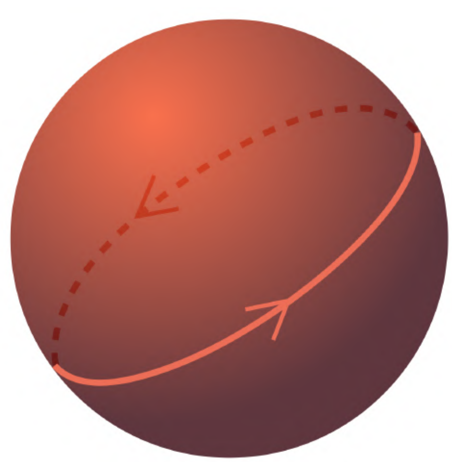
C_1 here is interpreted as a *constant angular velocity* $\omega = C_1$.

$$\omega = C_1$$



$$\therefore \quad \boxed{\phi(t) = C_1\, t}$$

This solution is a great circle.

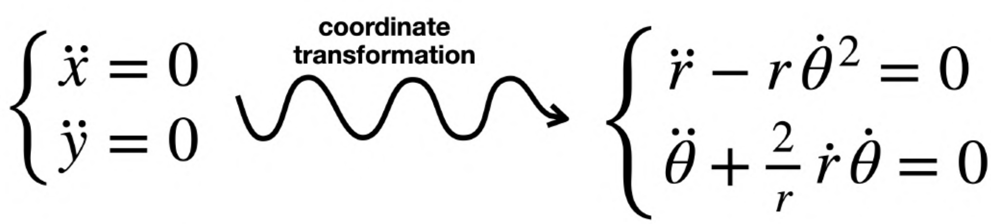


4 Coordinate Transformations

The idea here is that sometimes the geodesic equations look very messy in one coordinate system, but they simplify drastically in another one. So, instead of suffering through trying to solve them directly, you can perform a clever change of coordinates.

Think of geodesic paths as geometric objects, in the sense that they depend only on *intrinsic properties* of the manifold (like all the quantities that can be derived from the metric, and including the metric itself). So, geodesics are independent of coordinates. Which is great news for us.

You can have the same geodesic equation, but when written in different coordinate systems, they look completely unalike, even though they're actually describing literally the same geometric object:

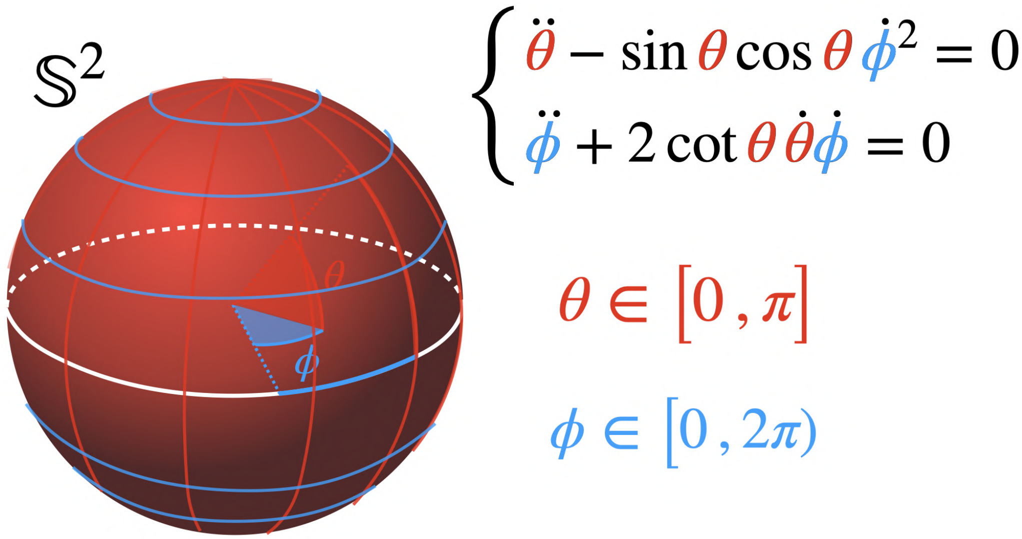


Just to illustrate it here, let's do this with the sphere $\mathbb{S}^2$ and show the same geodesics (which correspond to great circles) in 3 different intrinsic coordinate systems:

(I) Standard spherical coordinates $(\theta\,,\phi)$

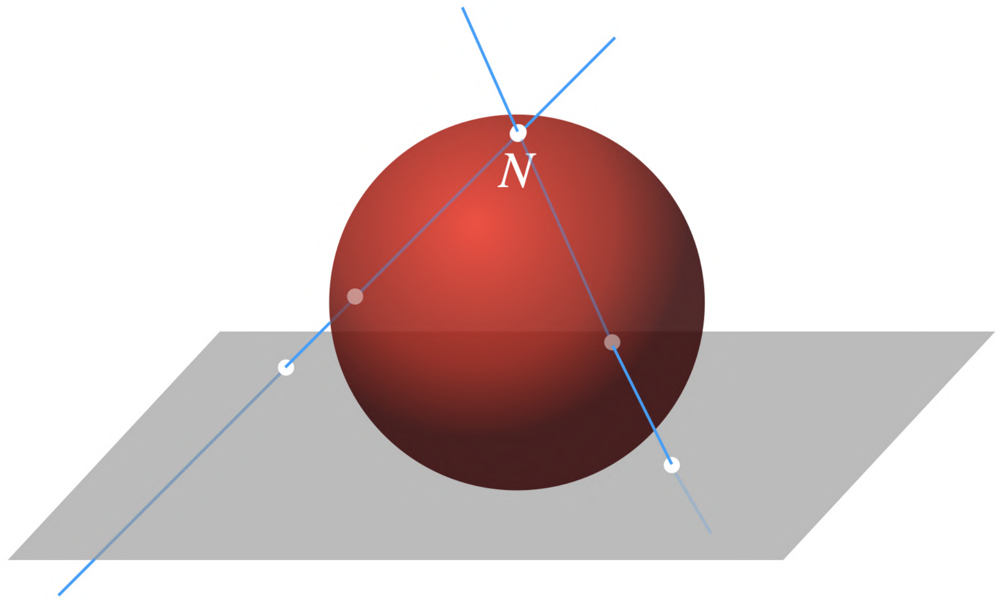
These are the coordinates we've seen earlier. The metric is $\boxed{ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2}$, and the geodesic equations are:

$$\begin{cases} \ddot{\theta} - \sin\theta\ \cos\theta\, \dot{\phi}^2 = 0 \\ \ddot{\phi} + 2\cot\theta\, \dot{\theta}\, \dot{\phi} = 0 \end{cases}$$



(II) Stereographic coordinates $(u\,,v)$

Now, we project the sphere from the north pole onto this plane (below), which is, nonetheless, an intrinsic mapping, or *chart* to be more precise.



The metric then is:

$$ds^2 = \frac{4}{(1+u^2+v^2)}\left(du^2+dv^2\right)$$

The geodesic equations, then, are:

$$\begin{cases} \ddot{u} - \frac{2u}{1+u^2+v^2}\left(\dot{u}^2+\dot{v}^2\right) = 0 \\ \ddot{v} - \frac{2v}{1+u^2+v^2}\left(\dot{u}^2+\dot{v}^2\right) = 0 \end{cases}$$

(III) Isothermal coordinates (locally conformal)

Locally, every $2D$ Riemannian metric can be written in a conformally flat form.

Suggestion: *If you want to learn about conformal geometry, check out the video and PDF below:*

Why Spacetime isn't Always Curved

PDF link: *Conformal Geometry*

But simply put, conformal coordinates are coordinates where **angles are preserved**: If two curves meet at 90° in one coordinate description, they will still meet at 90° in the new one. What might change are the distances or lengths: things can look stretched or shrunk, but the shape of angles is kept intact.

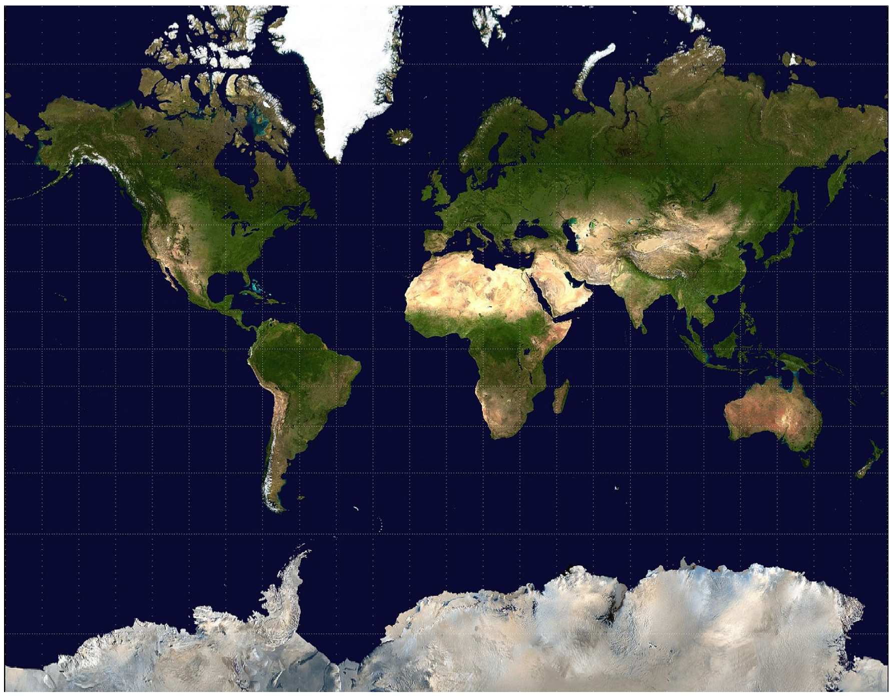
So, in our case the metric is:

$$\boxed{ds^2 = \Omega^2(u\,,v)\left(du^2 + dv^2\right)}$$

Where $\Omega(u\,,v)$ is a positive function called *conformal factor*.

In our case, we'll use the **Mercator projection**:

$$ds^2 = \frac{1}{\cosh^2 v} \left(du^2 + dv^2\right)$$



The Mercator metric tensor is:

$$g_{ij} = \frac{1}{\cosh^2 v} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

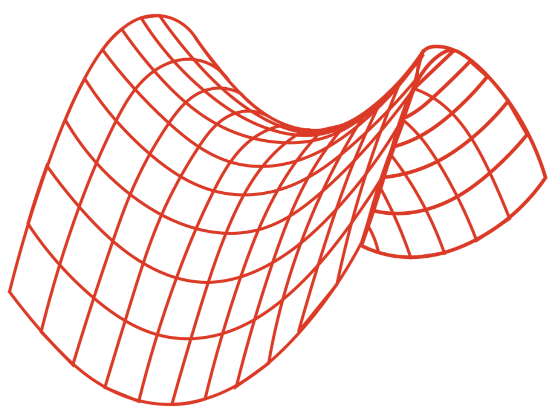
As a consequence, the geodesic equations become:

$$\begin{cases} \ddot{u} - 2\tanh(v)\,\dot{u}\,\dot{v} = 0 \\ \ddot{v} + \tanh(v)\,\dot{u}^2 - \tanh(v)\,\dot{v}^2 = 0 \end{cases}$$

Nice! Next, we'll see a concrete example of how this method allows us to simplify a set of very complicated geodesic **PDEs**, and potentially find explicit solutions.

The $2D$ **Hyperbolic Plane** $\mathbb{H}^2$:

$$\begin{cases} \text{(radial)} \quad \ddot{r} - \sinh(r)\,\cosh(r)\,\dot{\theta}^2 = 0 \\ \text{(angular)} \quad \ddot{\theta} + 2\coth(r)\,\dot{r}\,\dot{\theta} = 0 \end{cases}$$



This manifold is described through *hyperbolic geometry*. Of course, this saddle representation is not accurate for several reasons.

One of them is the fact that there is no ambient space in which the manifold is embedded. We want purely intrinsic description of it.

The second reason is that, as the name says, it's actually a plane, but with a metric defined in it that gives the appropriate curvature at each point. In this space, geodesics diverge exponentially.
We could very well use coordinates $(r\,,\theta)$ with the metric:

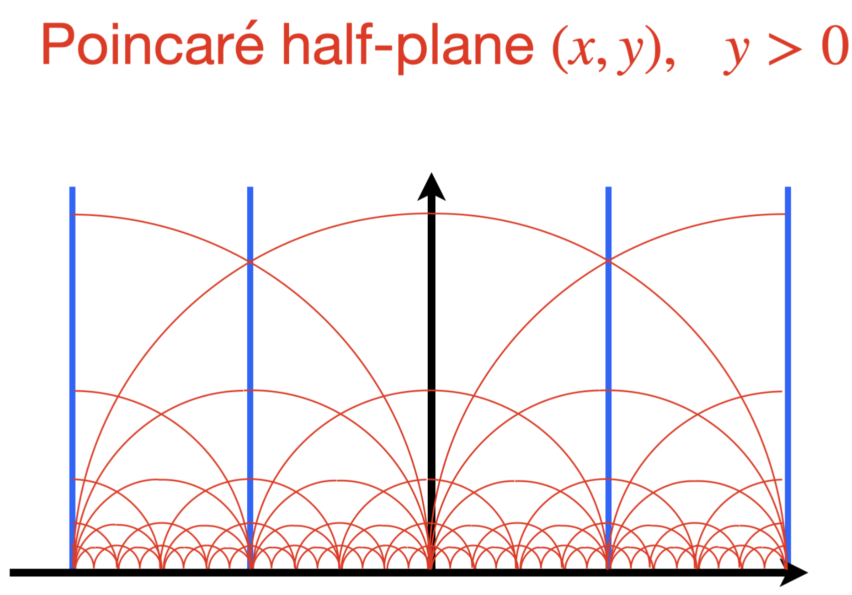
$$ds^2 = dr^2 + \sinh^2(r)\, d\theta^2$$

This choice would provide us with the following geodesic equations:

$$\begin{cases} \ddot{r} - \sinh(r)\, \cosh(r)\, \dot{\theta}^2 = 0 & (radial) \\ \ddot{\theta} + 2\coth(r)\, \dot{r}\, \dot{\theta} = 0 & (angular) \end{cases}$$

These are SUPER hard to solve!

Now, let's apply the method and transform coordinates to the **Poincaré half-plane coordinates** (x, y), with $y > 0$.



The metric is:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

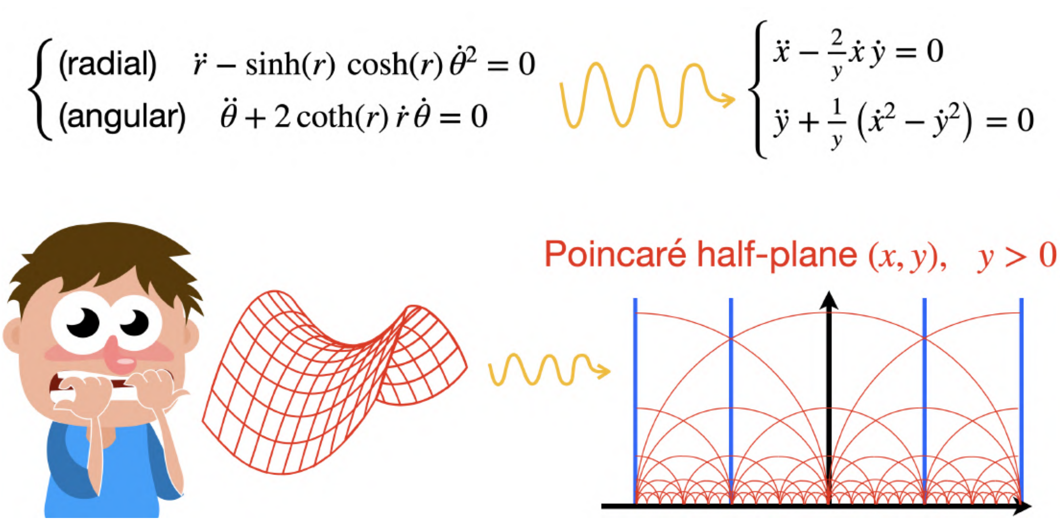
or

$$g_{ij} = \frac{1}{y^2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

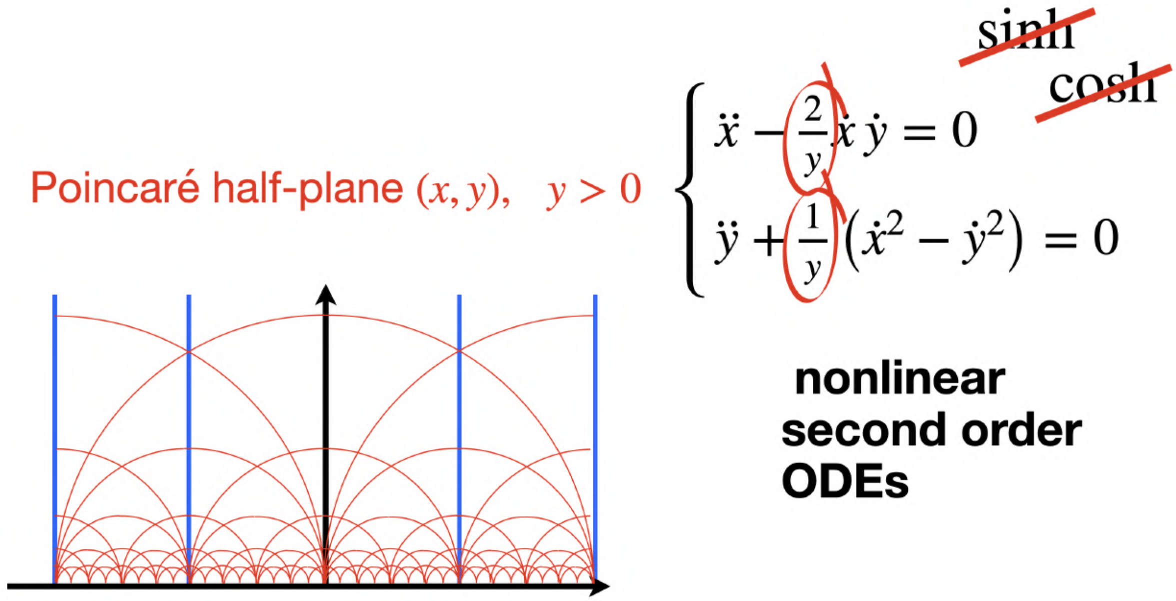
And the geodesic equations become (after calculating the Christoffel symbols):

$$\begin{cases} \ddot{x} - \frac{2}{y}\,\dot{x}\,\dot{y} = 0 \\ \ddot{y} + \frac{1}{y}\,\left(\dot{x}^2 - \dot{y}^2\right) = 0 \end{cases}$$

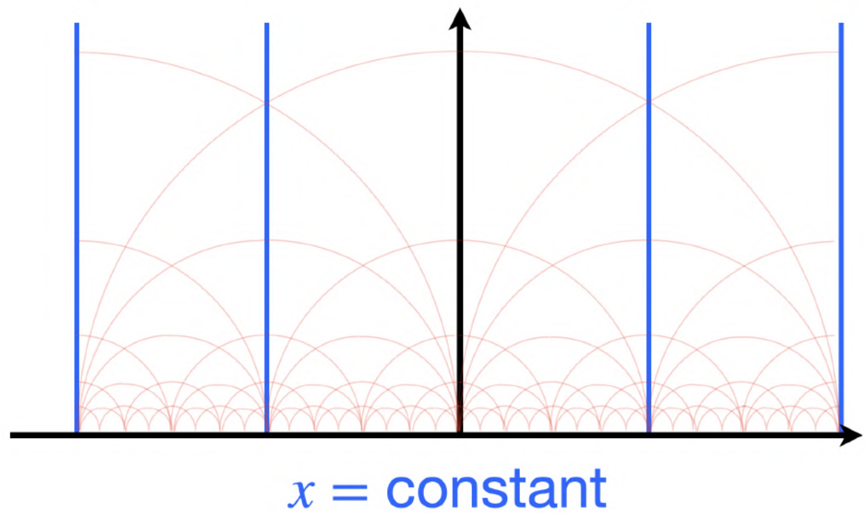
I know, this set of geodesic equations is not *trivial* at all.

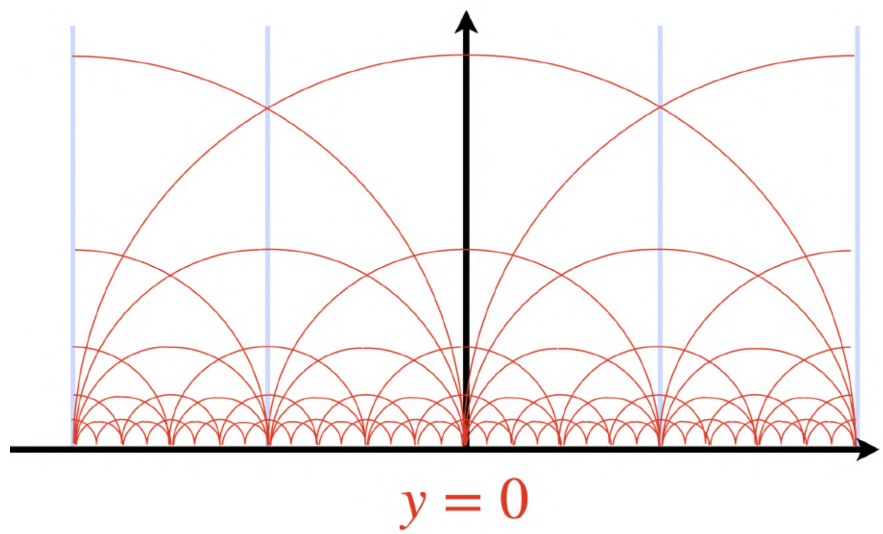


Just as before, we have a system of **nonlinear second-order ODEs**, but this time the structure is much simpler: there are only *rational functions* involved here (like $\frac{1}{y}$), no crazy things like sinh or cosh.



The geometry is much simpler as well: in the half-plane model, solutions are just vertical straight lines (x = constant) or semicircles orthogonal to the boundary (at $y = 0$).





5 Integrability & Quadrature

The **Quadrature method** is about avoiding the direct attack on the second-order geodesic ODEs (which are usually nonlinear and messy). Instead of working with the geodesic equations themselves, we notice that geodesics come from a variational principle: they are the curves that extremize the length functional ($\mathcal{S}[\gamma] = \int \sqrt{g_{ij}(x)\, \dot{x}^i\, \dot{x}^j}\, dt$). This means they can always be described by a Lagrangian built from the metric. Once we have this Lagrangian, we don't need to solve the ODEs directly.

Instead, we look for conserved quantities (like energy, momentum, angular momentum, etc) that come from symmetries of the system.

Energy E

$$2E = \dot{\theta}^2 + \frac{L^2}{\sin^2\theta}$$

Angular Momentum L

$$L = \sin^2\theta\,\dot{\phi}$$

Ok, so far this is pretty much the same as another method we've seen earlier. But here's the difference:

Once these conserved quantities are written down, the system can often be reduced to a *first-order equation*. At this point, we can rewrite it in the form:

$$\frac{d\theta}{dt} = F(\theta)$$

And then integrate it:

$$\int \frac{d\theta}{F(\theta)} = t + C\ , \quad (C = \text{constant})$$

That's the meaning of the word **quadrature** (in this context): *"to turn the problem into an integral"*.

The solution may not look simple, since it often involves *elliptic* (or other *special*) functions, but the point is that the problem is completely solved in principle, because the geodesic is fully determined by these integrals.

Before solving the problem that we're actually interested in, let's see a very simple example to get a hang on this method.

Consider the differential equation that describes a **simple harmonic oscillator**:

$$\boxed{\ddot{x} = -k\,x} \quad , \qquad (k = \text{constant})$$

Let's multiply both sides by $\dot{x}$:

$$\ddot{x} \cdot \dot{x} = -k\,x \cdot \dot{x}$$

Notice that the LHS $(\ddot{x} \cdot \dot{x})$ can be rewritten as $\frac{d}{dt}\left(\frac{1}{2}\,\dot{x}^2\right)$, and that the RHS $(-k\,x \cdot \dot{x})$ can be rewritten as $-\frac{d}{dt}\left(\frac{1}{2}\,k\,x^2\right)$:

$$\frac{d}{dt}\left(\frac{1}{2}\,\dot{x}^2\right) = -\frac{d}{dt}\left(\frac{1}{2}\,k\,x^2\right)$$

Now, we integrate everything with respect to t, and apply the **Fundamental Theorem of Calculus**:

$$\int \frac{d}{dt}\left(\frac{1}{2}\,\dot{x}^2\right)\,dt = -\int \frac{d}{dt}\left(\frac{1}{2}\,k\,x^2\right)\,dt \implies$$
$$\implies \frac{1}{2}\,\dot{x}^2 + \frac{1}{2}\,k\,x^2 = E$$

Here E is a constant, namely energy. And thus this equation is the mathematical expression of the **energy conservation law**.
The next step is to isolate $\dot{x}$:

$$\dot{x} = \sqrt{2E - k\,x^2}$$

And then we separate variables:

$$\frac{dx}{\sqrt{2E - k\,x^2}} = dt$$

This is the **quadrature form**, and the solution is given by an integral:

$$\int \frac{dx}{\sqrt{2E - k\,x^2}} = t + C \implies$$

$$\implies \int \frac{dx}{\sqrt{2E}\,\sqrt{1 - \frac{k}{2E}\,x^2}} = t + C \implies \left| \begin{matrix} u := \sqrt{\frac{k}{2E}}\,x \\ dx = \sqrt{\frac{2E}{k}}\,du \end{matrix} \right| \implies$$

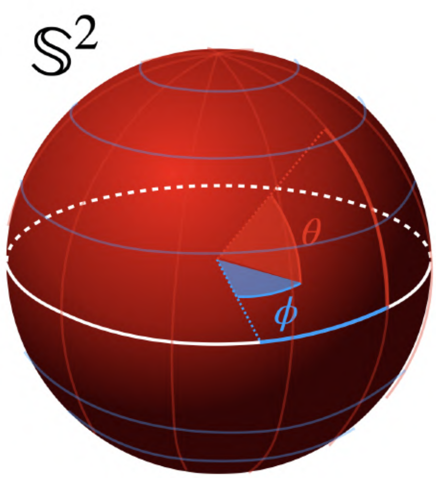
$$\implies \int \frac{\sqrt{\frac{2E}{k}}\,du}{\sqrt{2E}\,\sqrt{1-u^2}} = t + C \implies \frac{1}{\sqrt{k}} \int \frac{du}{\sqrt{1-u^2}} = t + C \implies$$

$$\text{(known from integral tables)} \quad \frac{1}{\sqrt{k}} \arcsin(u) + C' = t + C \implies$$

$$\implies \frac{1}{\sqrt{k}} \arcsin\left(\sqrt{\frac{k}{2E}}\,x\right) = t + (C - C') \implies$$

$$\implies \boxed{x(t) = \sqrt{\frac{2E}{k}} \sin\left(\sqrt{k}\,t + \varphi\right)} \qquad \varphi := (C - C') \cdot \sqrt{k} = \text{constant}$$

Nice! That's similar to what we want to do with our geodesic equation.



$$ds^2 = d\theta^2 + \sin^2\theta\, d\phi^2$$

$$g = \begin{bmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{bmatrix}$$

Again, in our sphere case, the metric is:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2$$

The energy conservation gives:

$$2E = \dot{\theta}^2 + \frac{L^2}{\sin^2\theta}$$

Rearranging it into quadrature form, we get:

$$\dot{\theta}^2 = 2E - \frac{L^2}{\sin^2\theta} \implies$$

$$\implies d\theta = \pm dt\sqrt{2E - \frac{L^2}{\sin^2(\theta)}} \implies$$

$$\implies \int \frac{d\theta}{\sqrt{2E - \frac{L^2}{\sin^2(\theta)}}} = \pm \int dt \implies$$

$$\implies \int \frac{\sin\theta\, d\theta}{\sqrt{2E\, \sin^2\theta - L^2}} = \pm t + C \implies$$

$$\left| \begin{matrix} u := \cos\theta \implies \sin^2\theta = 1 - u^2 \\ du = -\sin\theta\, d\theta \end{matrix} \right|$$

$$\implies -\int \frac{du}{\sqrt{2E(1-u^2) - L^2}} = \pm t + C \implies$$

$$\implies \int \frac{du}{\sqrt{(2E - L^2) - 2Eu^2)}} = -(\pm t + C) \implies$$

$$\implies \int \frac{du}{\sqrt{2E - L^2}\sqrt{1 - \left(\sqrt{\frac{2E}{2E-L^2}}\, u\right)^2}} = \pm t - C \implies$$

$$\left| \begin{array}{c} y := \sqrt{\frac{2E}{2E-L^2}}\, u \\ dy = \sqrt{\frac{2E}{2E-L^2}}\, du \quad \Longrightarrow \quad du = \sqrt{\frac{2E-L^2}{2E}}\, dy \end{array} \right|$$

$$\Longrightarrow \quad \frac{1}{\sqrt{2E-L^2}} \int \frac{\sqrt{\frac{2E-L^2}{2E}}\, dy}{\sqrt{1-y^2}} = \pm t - C \quad \Longrightarrow$$

$$\Longrightarrow \quad \frac{1}{\sqrt{2E-L^2}} \cdot \frac{\sqrt{2E-L^2}}{\sqrt{2E}} \int \frac{dy}{\sqrt{1-y^2}} = \pm t - C \quad \Longrightarrow$$

$$\Longrightarrow \quad \frac{1}{\sqrt{2E}} \arcsin(y) = \pm t - C \quad \Longrightarrow$$

$$\Longrightarrow \quad \frac{1}{\sqrt{2E}} \arcsin\left(\sqrt{\frac{2E}{2E-L^2}}\, u \right) = \pm t - C \quad \Longrightarrow$$

$$\Longrightarrow \quad \frac{1}{\sqrt{2E}} \arcsin\left(\sqrt{\frac{2E}{2E-L^2}}\, \cos\theta \right) = \pm t - C \quad \Longrightarrow$$

$$\Longrightarrow \quad \sqrt{\frac{2E}{2E-L^2}}\, \cos\theta \;=\; \sin\left(\sqrt{2E}\, (\pm t - C) \right) \quad \Longrightarrow$$

$$\Longrightarrow \quad \boxed{\theta(t) \;=\; \arccos\left(\sqrt{\frac{2E-L^2}{2E}}\, \sin\left(\pm\sqrt{2E}\, t \;+\; \varphi_0 \right) \right)} \qquad \varphi_0 := -\sqrt{2E}\, C$$

That's the first one. In order to find the second, we use a similar approach:

$$\dot{\phi} = \frac{L}{\sin^2(\theta(t))} \quad \Longrightarrow$$

$$\Longrightarrow \quad \int d\phi = \int \frac{L}{\sin^2(\theta(t))}\, dt \quad \Longrightarrow$$

$$\implies \quad \phi(t) = \int \frac{L}{\sin^2\left(\arccos\left(\sqrt{\frac{2E-L^2}{2E}}\,\sin\left(\pm\sqrt{2E}\,t\,+\,\varphi_0\right)\right)\right)}\,dt\;=$$

$$= L\int \frac{L}{1-\cos^2\left(\arccos\left(\sqrt{\frac{2E-L^2}{2E}}\,\sin\left(\pm\sqrt{2E}\,t\,+\,\varphi_0\right)\right)\right)}\,dt\;=$$

$$= L\int \frac{L}{1-\frac{(2E-L^2)}{2E}\,\sin^2\left(\pm\sqrt{2E}\,t\,+\,\varphi_0\right)}\,dt\;=$$

$$\left|\begin{array}{c} u := \cot\left(\pm\sqrt{2E}\,t+\varphi_0\right) \;\implies\; \sin^2\left(\pm\sqrt{2E}\,t+\varphi_0\right) = \frac{1}{1+u^2} \\ du = \pm\sqrt{2E}\,\csc^2\left(\pm\sqrt{2E}\,t+\varphi_0\right)\,dt = \frac{\pm\sqrt{2E}\,dt}{\sin^2\left(\pm\sqrt{2E}\,t+\varphi_0\right)} = \frac{\pm\sqrt{2E}\,dt}{\frac{1}{1+u^2}} \;\implies \\ \implies\; dt = \frac{du}{\pm\sqrt{2E}\,(1+u^2)} \end{array}\right|$$

$$= -\frac{L}{\sqrt{2E}}\int \frac{du}{\left(1-\frac{(2E-L^2)}{2E}\cdot\frac{1}{1+u^2}\right)\cdot(1+u^2)} =$$

$$= -\frac{L}{\sqrt{2E}}\int \frac{du}{\left(1+\frac{-2E+L^2}{2E}\right)+u^2} =$$

$$= -\frac{L}{\sqrt{2E}}\int \frac{du}{\left(\frac{2E-2E+L^2}{2E}\right)+u^2} =$$

$$= -\frac{L}{\sqrt{2E}}\int \frac{du}{\frac{L^2}{2E}\left[1+\left(\frac{\sqrt{2E}}{L}\,u\right)^2\right]} =$$

$$= -\frac{L}{\sqrt{2E}}\cdot\frac{2E}{L^2}\int \frac{du}{1+\left(\frac{\sqrt{2E}}{L}\,u\right)^2} =$$

$$= -\frac{\sqrt{2E}}{L}\int \frac{du}{1+\left(\frac{\sqrt{2E}}{L}\,u\right)^2} =$$

$$\left| \begin{array}{c} z := \frac{\sqrt{2E}}{L} u \implies u = \frac{L}{\sqrt{2E}} z \\ du = \frac{L}{\sqrt{2E}} dz \end{array} \right|$$

$$= -\frac{\sqrt{2E}}{L} \cdot \frac{L}{\sqrt{2E}} \int \frac{dz}{1+z^2} = -\arctan(z) = -\arctan\left(\frac{\sqrt{2E}}{L} u\right) \implies$$

$$\implies \boxed{\phi(t) = -\arctan\left(\frac{\sqrt{2E}}{L} \cot\left(\pm\sqrt{2E}\, t + \varphi_0\right)\right)}$$

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