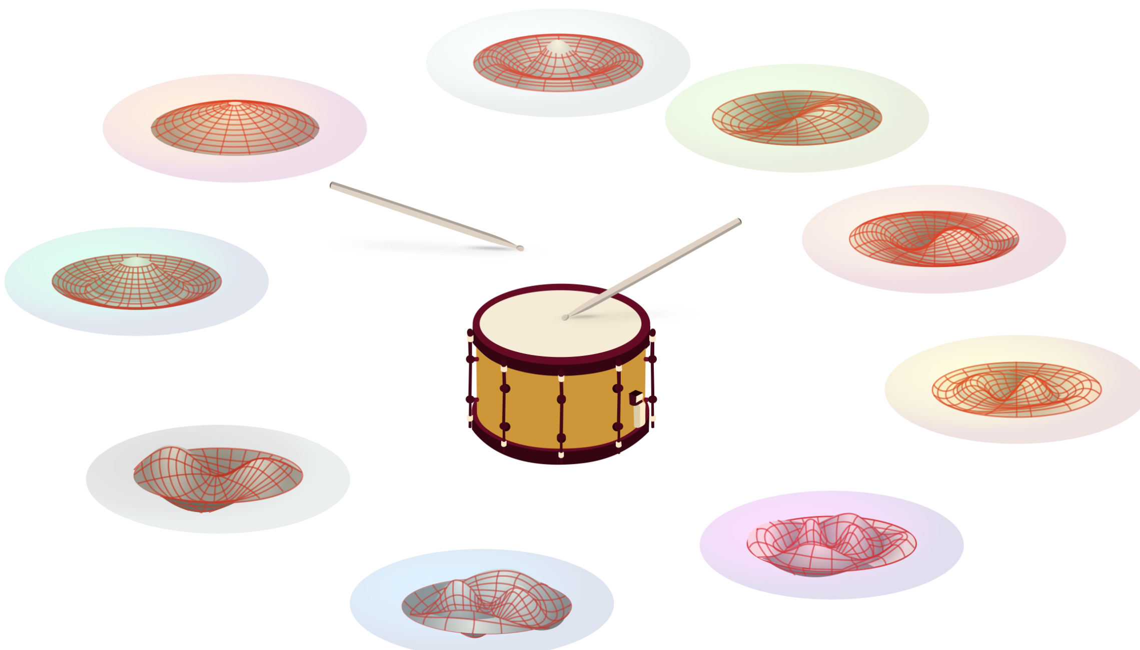
Can You Hear the Shape of a Drum?

by DiBeos

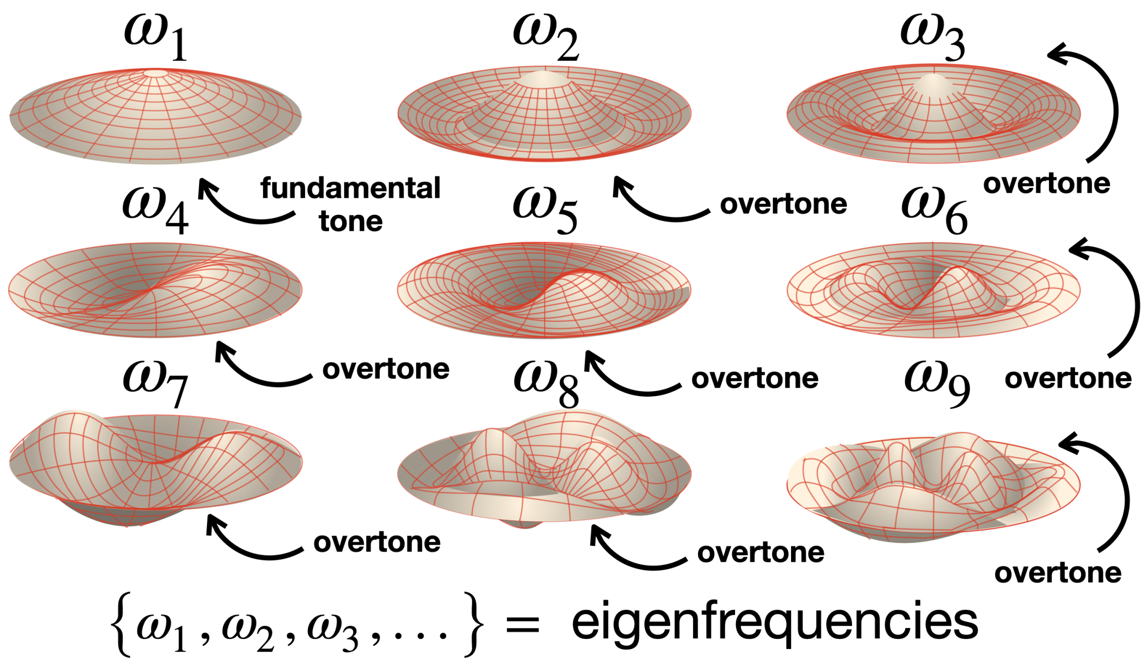


"The power of mathematics is often to change one thing into another, to change geometry into language." – **Marcus du Sautoy**

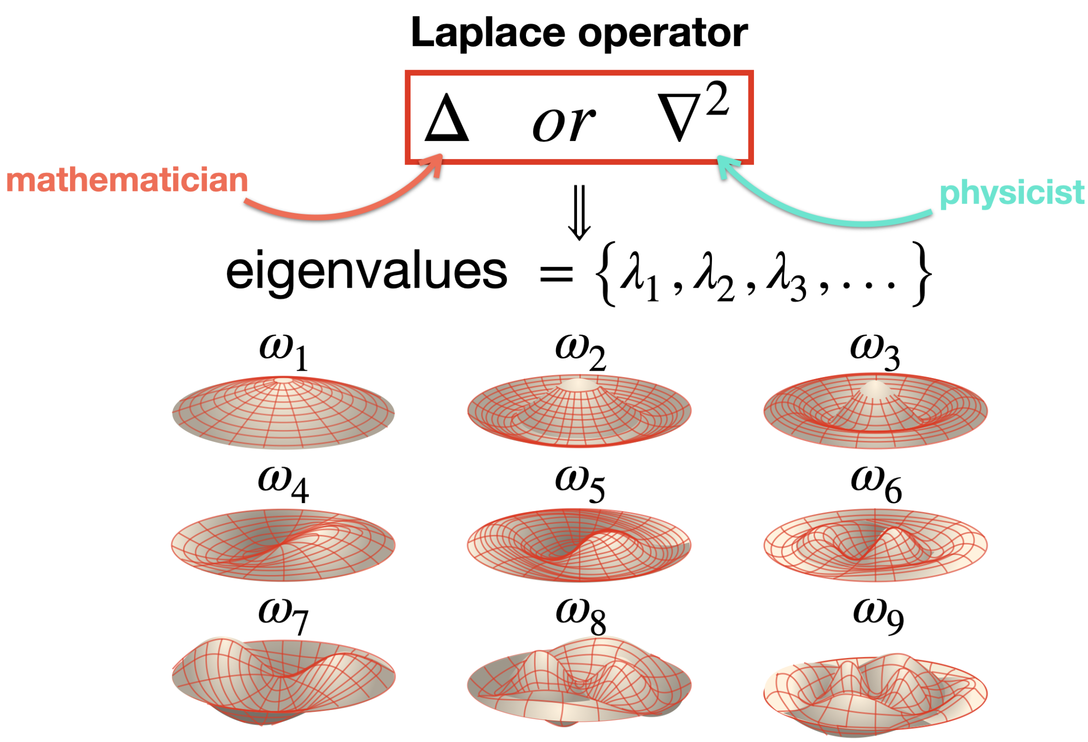
Introduction

Imagine striking a drumhead and listening to the sound of each strike. The vibration of the membrane is directly guided by extremely precise math.

Each oscillation has a specific frequency, and together they form a sequence of **eigenfrequencies**: the lowest of which is the *fundamental tone*, and the highest of which are the *overtones*.

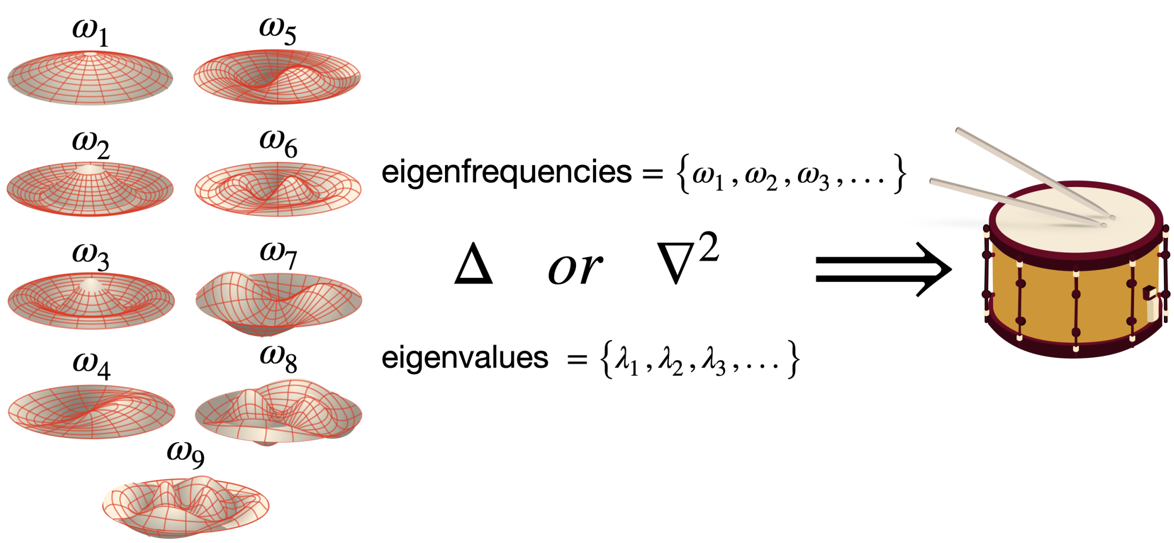


Mathematically speaking, each of these modes have a direct correspondence to an eigenvalue of the **Laplace operator**.

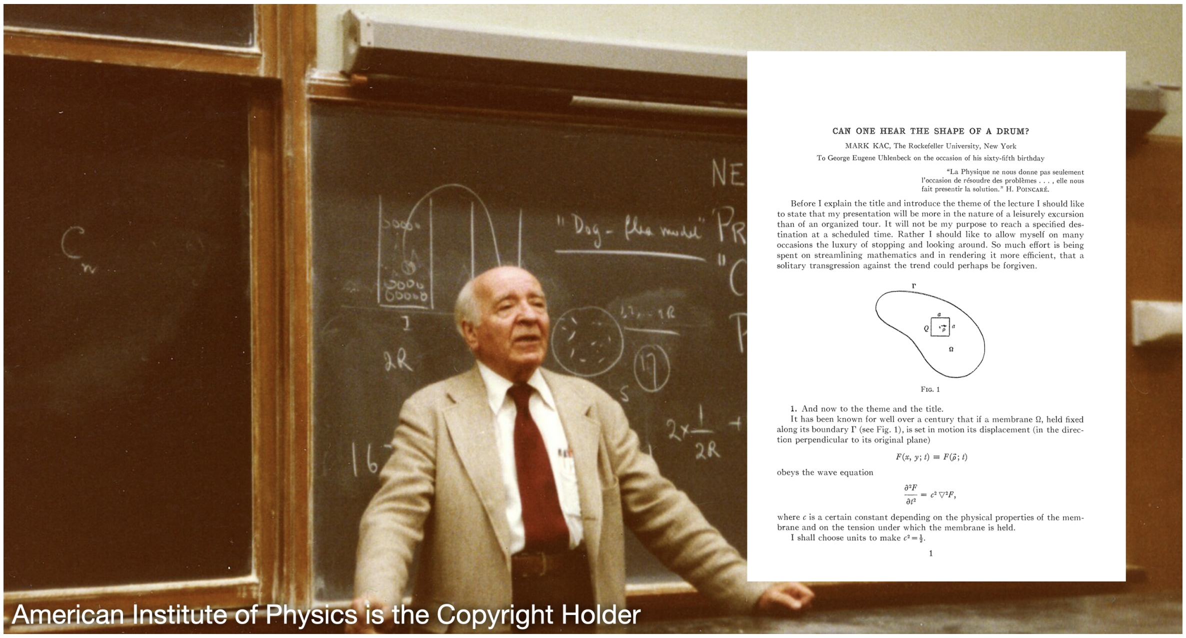


Putting all of this together gives us the complete description of the drum's vibrations.

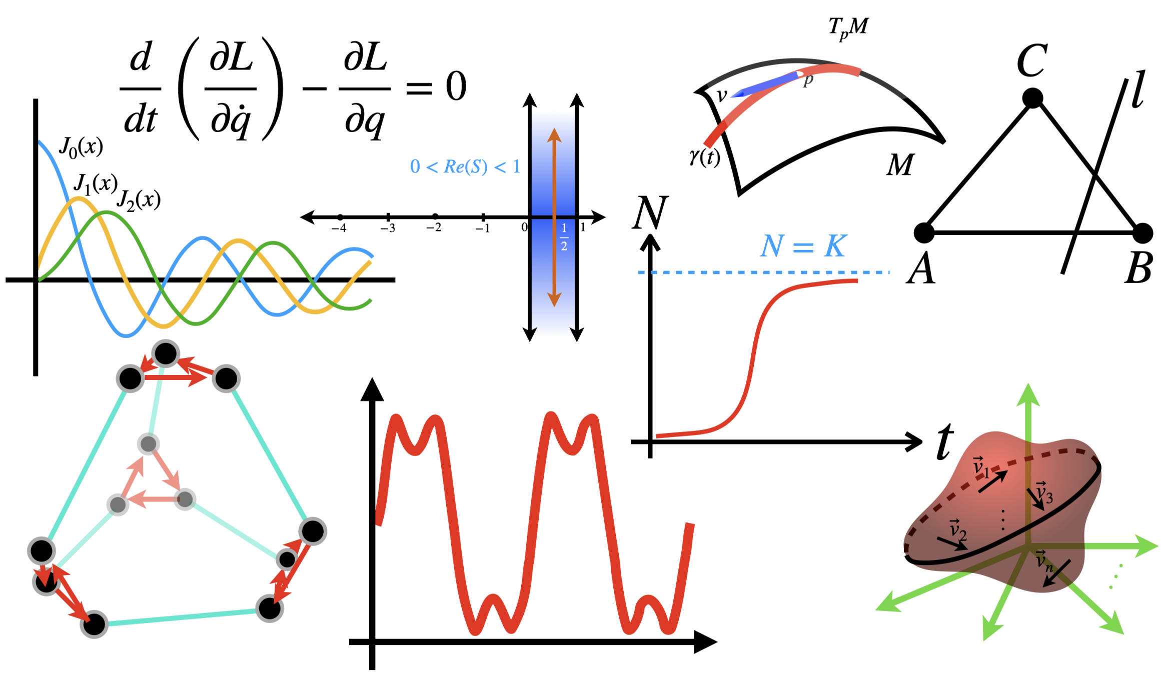
Now, here's a very interesting thought experiment: if we had the entire spectrum, every frequency, every overtone, could we then (in principle), reconstruct the exact shape of the drumhead itself? So basically: could we transform pure sound directly into geometry?



This was the question that mathematician Mark Kac asked in his famous 1966 paper titled *"Can One Hear the Shape of a Drum"*, and the answer to this might be pretty surprising.



But first, to properly, and deeply study this problem, we need to take a look at many, and I mean many, different fields of pure math, which includes things like **Partial Differential Equations (PDEs)**, **Spectral Theory**, **Fourier Analysis**, **Geometry**, **Group Theory**, **Asymptotic Analysis**, **Linear Algebra** and **Complex Analysis with Special Functions**.

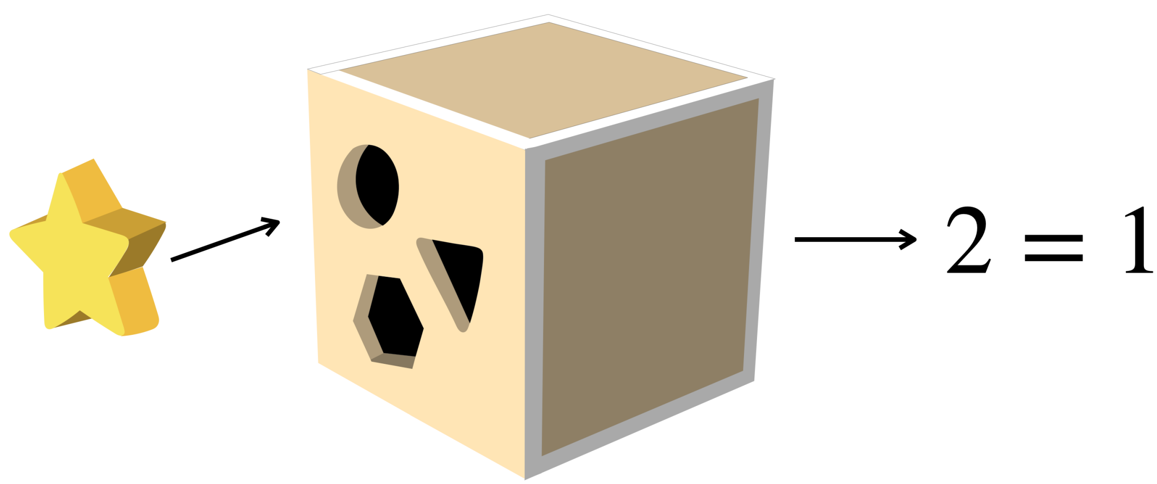


But don't feel discouraged or overwhelmed. We won't see all of these areas here. And, honestly, you don't need to know every detail of all of them. What you do need is just the right amount of insight from each of these in order to answer this beautiful question of whether we can, or cannot, "hear" the shape of a drum.

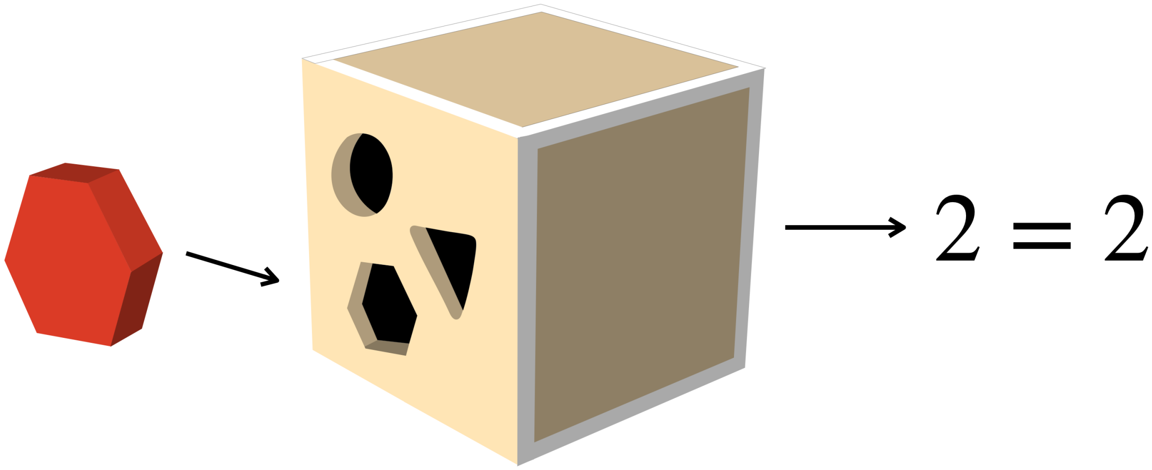
The Wave Equation

Let's go ahead and start with PDEs:

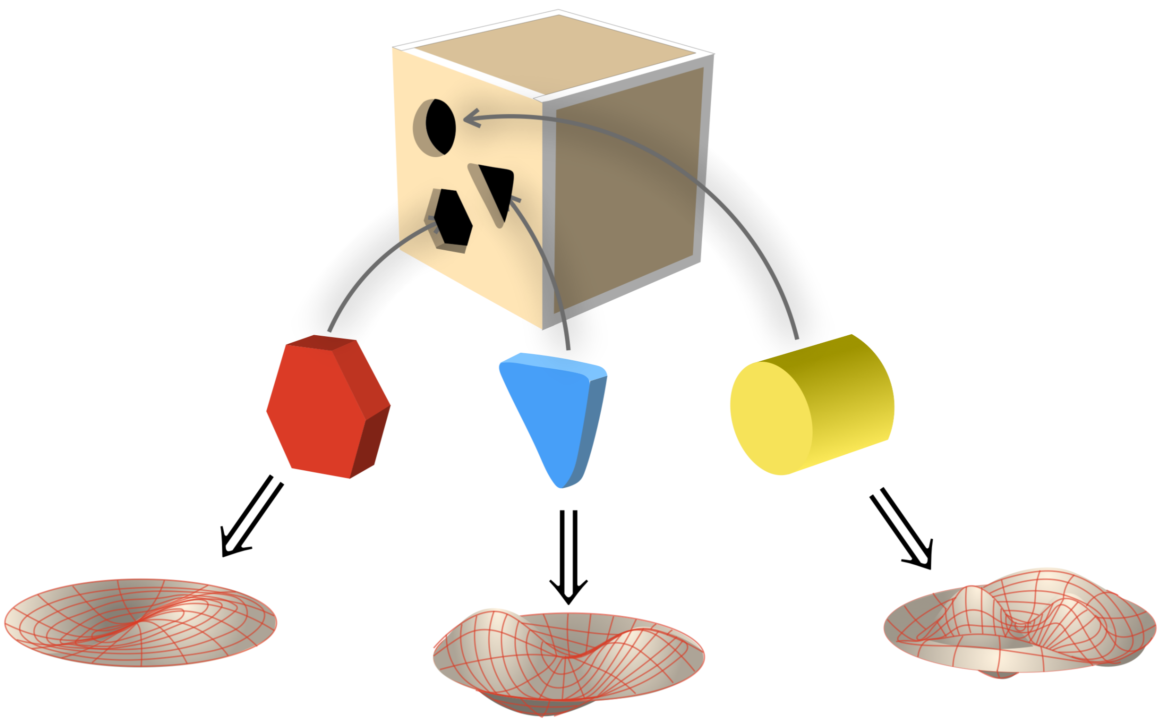
I like to think about PDEs as *machines with entry points*. That is, if you try to fit the wrong object into the machine (so, the wrong solution), it doesn't work and the machine will spit out a contradiction, like $2 = 1$.



But if you happen to be lucky, or be clever enough, to find the right object that fits into the entry point (so, the correct solution), then you strike gold, and the machine produces a true statement, let's say something like $2 = 2$.

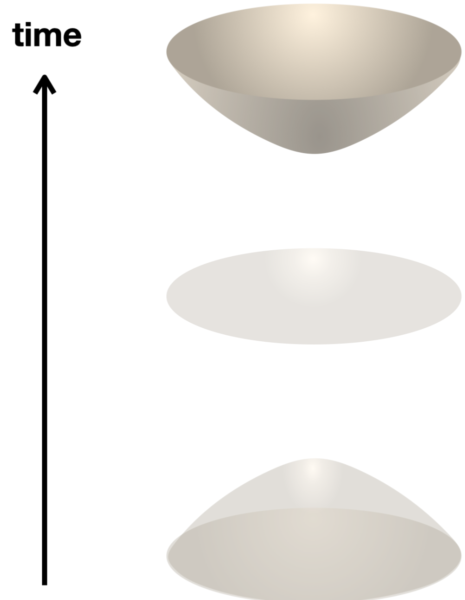


It means that the solution you found is very precious, because it effectively describes the phenomena you're trying to model.

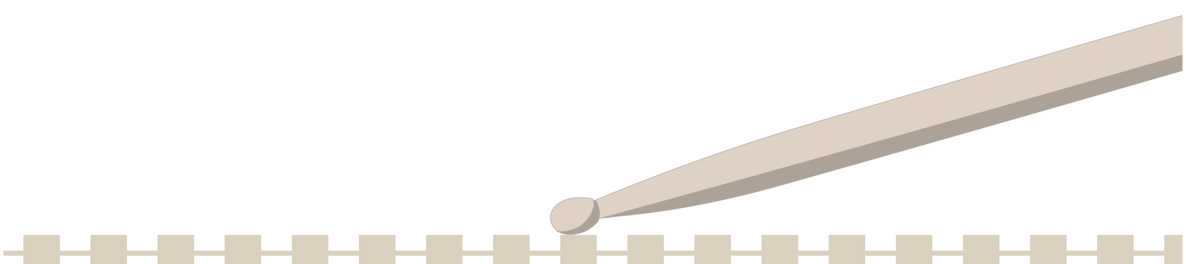


In our particular problem, the PDE we have is what describes how quantities change in space and time when those changes are linked together.

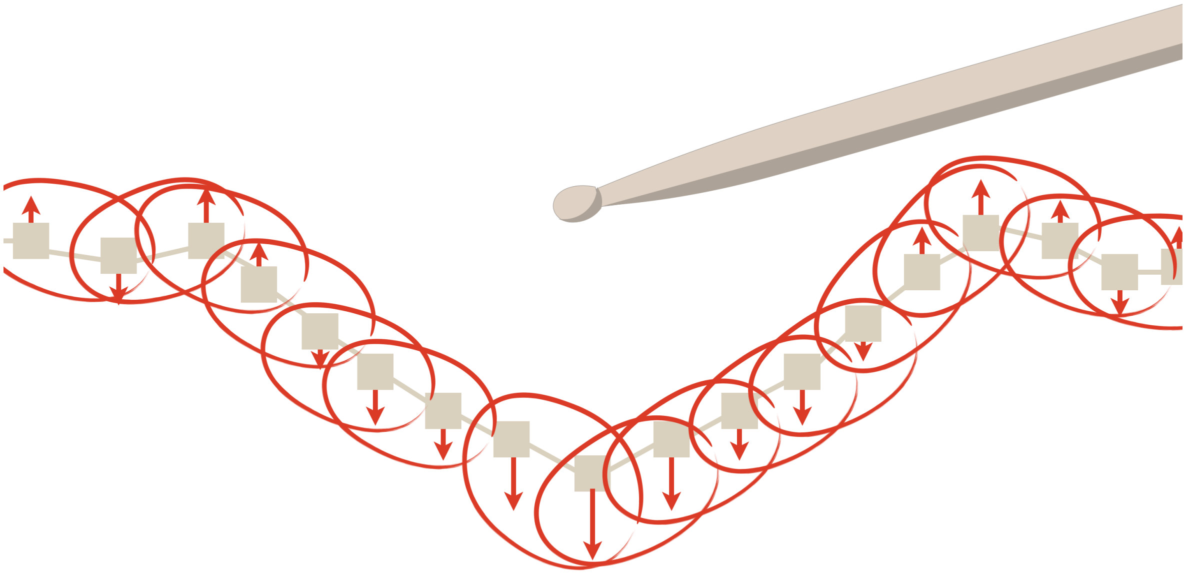
PDE: $$\frac{\partial^2 u}{\partial t^2} = c^2 \, \Delta u$$



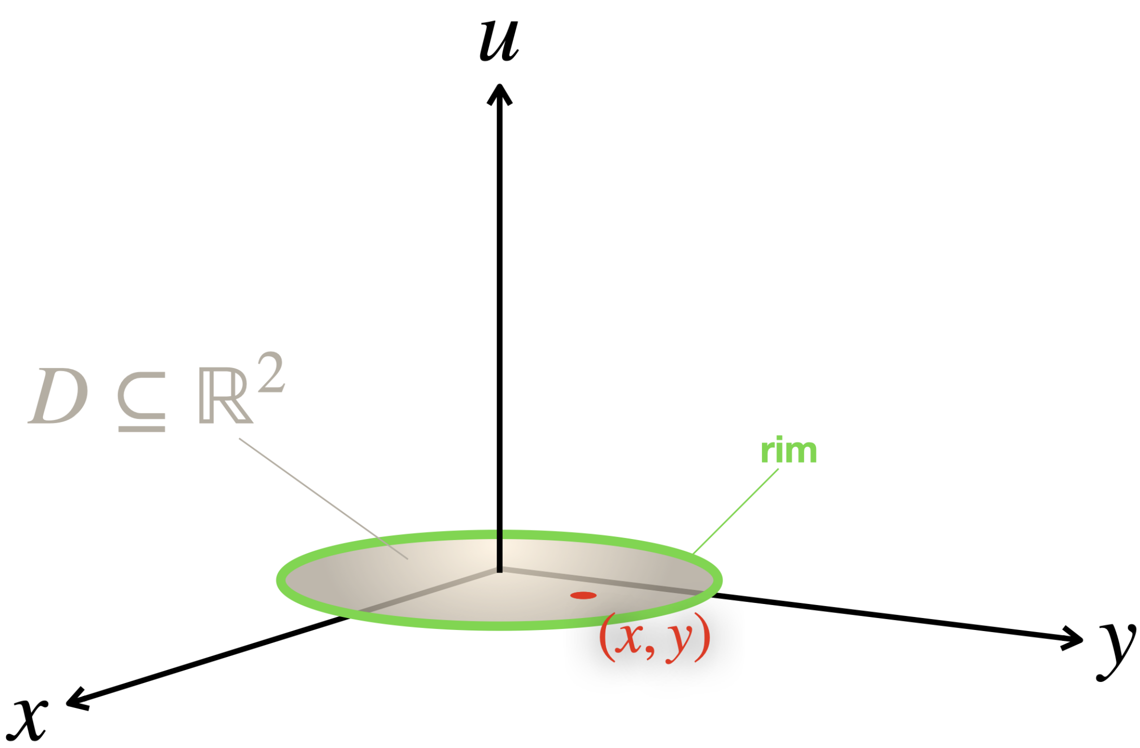
The stretched drumhead is a thin elastic membrane.



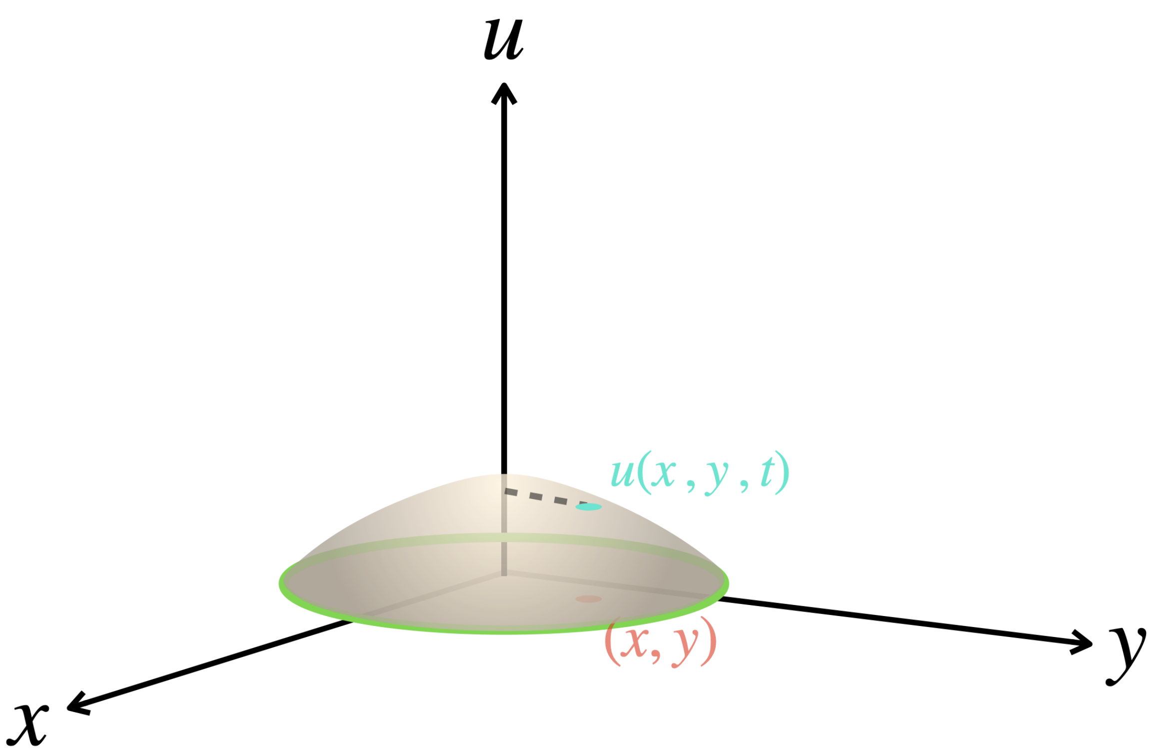
When you strike it, every point moves up and down, but they don't do so independently, because each point is influenced by its neighbors through *tension*.



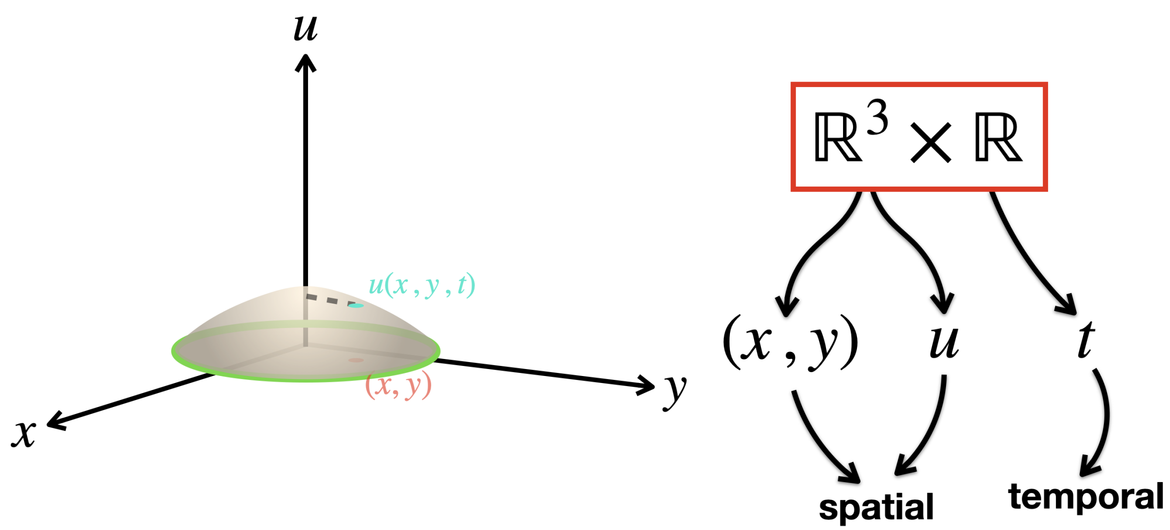
The drumhead is modeled as a 2D domain $D \subset \mathbb{R}^2$, lying flat in the plane of the rim. (x, y) specifies a point on this 2D membrane.



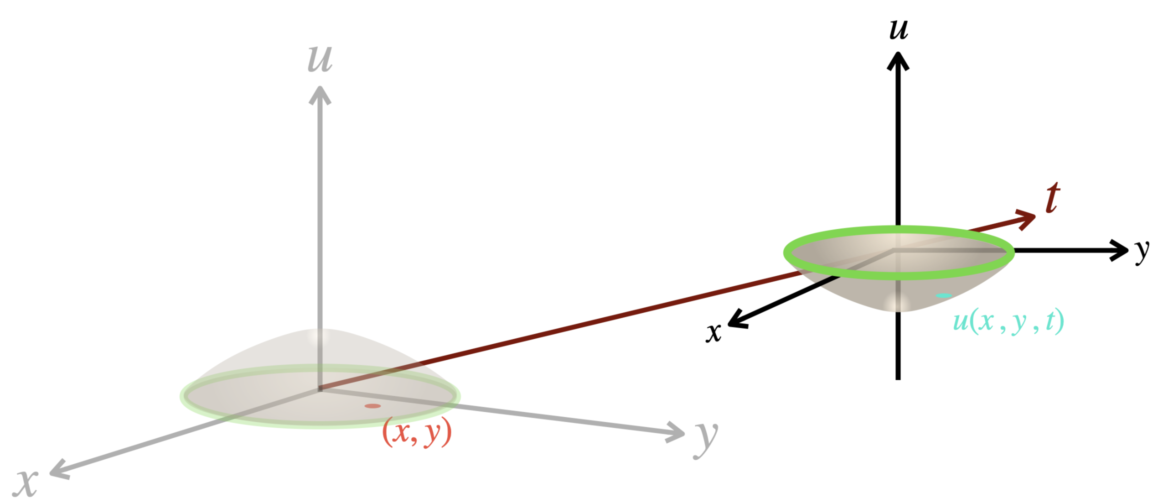
$u(x, y, t)$ gives the vertical displacement (or "height") of that point above or below the rest position, for each time t.


u
$u(x, y, t)$
(x, y)
x
y

Altogether, the motion lives in $\mathbb{R}^3 \times \mathbb{R}$: two spatial coordinates for the membrane, one extra vertical dimension for displacement, and one for time.



So you can think of it as a surface oscillating in 3D space, evolving along the time axis.



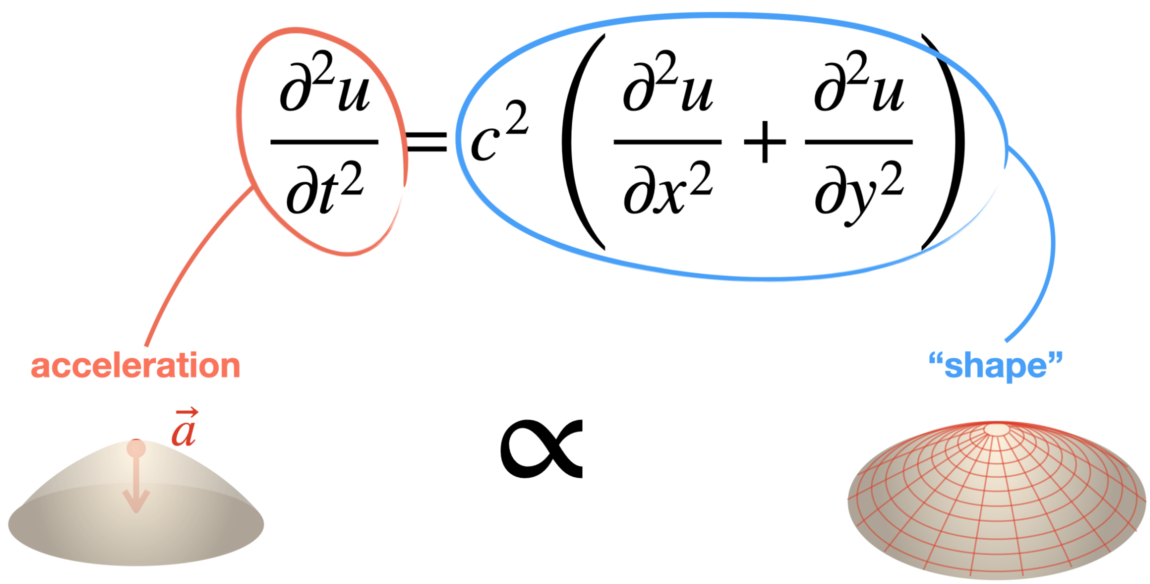
(Of course, trying to draw this 4D motion is kind of awkward, but I hope you get the idea)

Note: *If you'd like to be the first to find out when we launch our very first books and courses, sign up with your email address on our homepage, dibeos.net.*

The motion $u(x\,,\,y\,,\,t)$ of the membrane satisfies the wave equation:

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)}$$

Here, c is the *wave speed*. This equation balances how the membrane *accelerates* in time (left-hand side) with how it *bends in space* (right-hand side). The acceleration and the geometry of points of the membrane are not equal, but are directly proportional by the factor c^2.



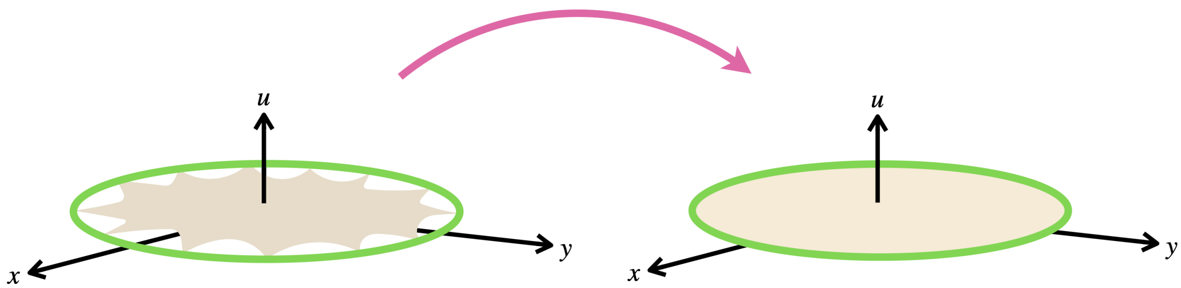
But to make it actually a drum, we add the following physical condition:

$$\boxed{u(x\,,\,y\,,\,t) = 0} \quad , \qquad \forall (x,y) \in \partial D \text{ (boundary of D)}$$

This is known as the **Dirichlet boundary condition**, and it imposes that every point on the membrane's boundary is fixed at height zero at all times. Physically speaking this is the rim of the drum being clamped so it cannot move.

Dirichlet boundary condition

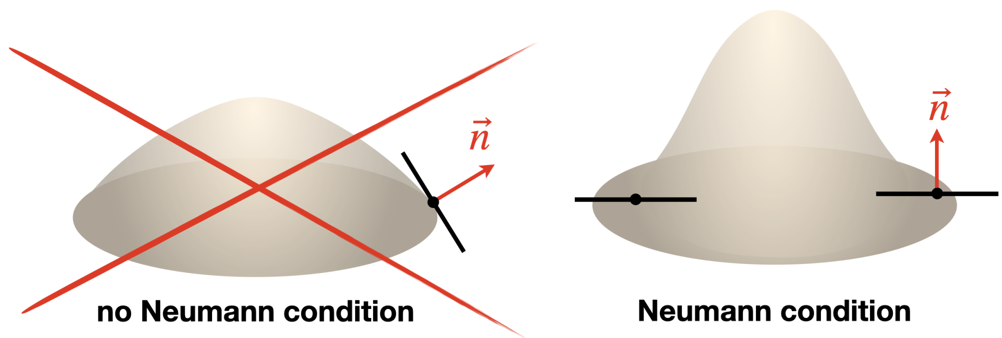
$$\boxed{u(x\,,y\,,t) = 0 \quad , \quad \forall (x\,,y) \in \partial D}$$



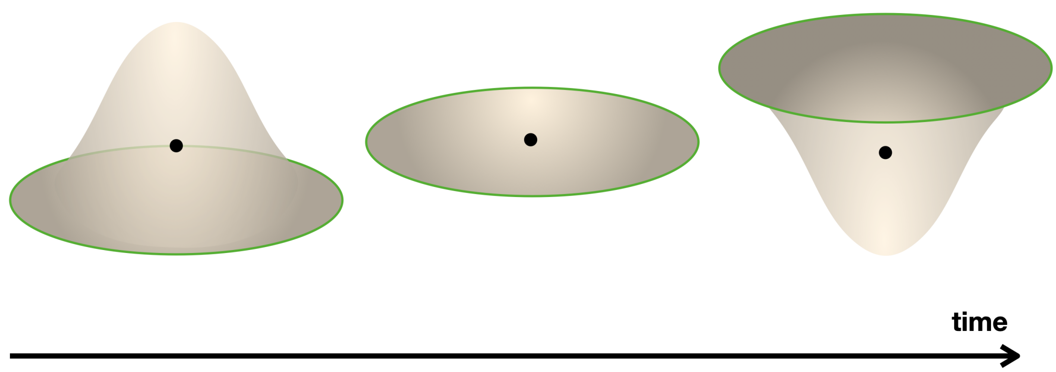
Let's contrast it with the **Neumann boundary condition**. This condition would impose that the normal derivative vanishes $\left(\frac{\partial u}{\partial n} = 0\right)$ on the boundary ∂D, in which case the membrane has no slope in the normal direction at the boundary.

Neumann boundary condition

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial D$$



Physically, this condition alone corresponds to a free edge that can move up and down but doesn't bend at the edge. That's not exactly how drums work...



And that's why we'll stick to the Dirichlet condition only.

In order to analyze vibrations, we look for *stationary wave solutions*.

$$\boxed{\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)} \implies \frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) u \implies$$

$$\implies \boxed{\frac{\partial^2 u}{\partial t^2} = c^2\,\Delta u}$$

Let's try to find solutions with a method called "separation of variables".

We assume that there are solutions of the form $u(x\,,\,y\,,\,t) := \Phi(x\,,\,y)\,T(t)$, i.e. the function u (which depends on the spatial $(x\,,\,y)$ and on the temporal t components) can be separated into a function Φ that depends only on the position of the point on the membrane, and a function T that depends only on the time evolution.

$$u(x\,,y\,,t) = \overbrace{\Phi}^{\text{position}}\underbrace{(x\,,y)}_{\text{spatial}}\overbrace{T}^{\text{time}}\underbrace{(t)}_{\text{temporal}}$$

This is a very strong assumption, by the way. But it's valid in this case, because the wave equation is linear (i.e. u and its derivatives only appear to the first power, and they're not multiplied together: $\nexists\; u^2\,,\,uu'\,,\,(\nabla u)^2$). Furthermore, its coefficient $\left(c^2\right)$ is time-independent, and the boundary condition is homogeneous and time-independent as well.

Anyway, let's plug $u(x\,,\,y\,,\,t) := \Phi(x\,,\,y)\,T(t)$ into the PDE:

$$\frac{\partial^2}{\partial t^2}\,(\Phi(x\,,\,y)\,T(t)) = c^2\left[\frac{\partial^2}{\partial x^2}\,(\Phi(x\,,\,y)\,T(t)) + \frac{\partial^2}{\partial y^2}\,(\Phi(x\,,\,y)\,T(t))\right] \implies$$

$$\implies \quad \Phi \, \frac{\partial^2 T}{\partial t^2} = c^2 \left(\frac{\partial^2 \Phi}{\partial x^2} \, T + \frac{\partial^2 \Phi}{\partial y^2} \, T \right)$$

Dividing both sides by $c^2 \, \Phi T$, such that $\Phi \neq 0$ and $T \neq 0$ for points not in the boundary, we get:

$$\frac{1}{c^2 \, T} \cdot \frac{\partial^2 T}{\partial t^2} = \left(\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right) \frac{1}{\Phi} \quad \implies$$

$$\implies \quad \boxed{\frac{T''(t)}{c^2 \, T(t)} = \frac{\Delta \Phi(x\,,\,y)}{\Phi(x\,,\,y)}}$$

Since the left-hand side only depends on t, the right-hand side only depends on x and y, and they are equal ("=" between them), then the whole thing is just constant:

$$\frac{T''}{c^2 \, T} = \frac{\Delta \Phi}{\Phi} =: -\lambda \qquad (\lambda \geq 0 \;\; \text{constant}) \quad \implies$$

$$\implies \quad (\, I \,) \begin{cases} \Delta \Phi + \lambda \, \Phi = 0 \quad , & (x\,,\,y) \in D \\ \Phi = 0 \quad , & (x\,,\,y) \in \partial D \;\; \text{(Dirichlet condition)} \end{cases}$$

$$\implies \quad (\, II \,) \; T'' + \omega^2 T = 0 \quad , \;\; \text{where } \omega^2 := c^2 \, \lambda$$

Notice that $(\, II \,)$ is an ODE (ordinary differential equation), and it's not difficult to solve, since this is the famous equation for a *simple harmonic oscillator*. I mean, it does make sense that we bumped into it, since the motion is clearly periodic!

Solutions for the temporal part $T(t)$ take the following form:

$$T(t) = A \, \cos(\omega t) + B \, \sin(\omega t)$$

(Check that this is indeed a solution $T(t)$ by inserting it in the equation $(\, II \,)$)

constants
(initial condition: $T(0)$ and $T'(0)$)

$$T(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$\boxed{\omega := c\sqrt{\lambda}}$$

angular frequency

Here, A and B are constants determined by the initial conditions of the problem, namely the *initial displacement* $T(0)$ and *initial velocity* $T'(0)$. Since we haven't specified those conditions explicitly, we just leave A and B as free constants.

And therefore, each solution takes the following form:

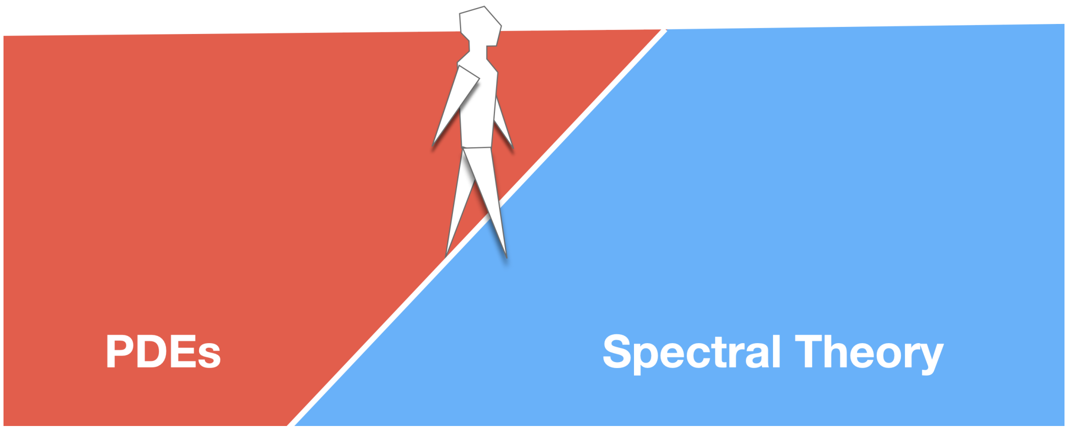
$$\boxed{u(x\,,\,y\,,\,t) = \Phi(x\,,\,y)\ \left(A\,\cos(\omega t) + B\,\sin(\omega t)\right)}$$

Ok... but what about an explicit expression for the spatial function $\Phi(x\,,\,y)$ of the motion?

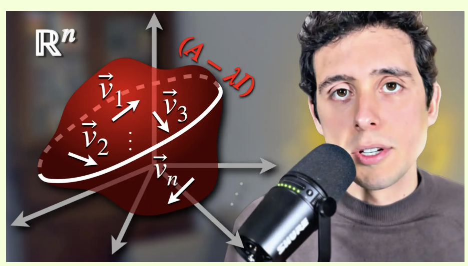
$$(\,I\,)\ \begin{cases} \Delta\Phi + \lambda\,\Phi = 0 \quad , & (x\,,\,y) \in D \\ \Phi = 0 \quad , & (x\,,\,y) \in \partial D \ \ \text{(Dirichlet condition)} \end{cases}$$

$(\,I\,)$ is an *eigenvalue problem* for the Laplace operator Δ.

As you can see, we're now starting to cross the line from the mathematical area of **PDEs** into **Spectral Theory**.



Suggestion: *If you want to strengthen your understanding of* ***eigenvectors & eigenvalues****, check out the video and PDF below, where we explain them in detail.*

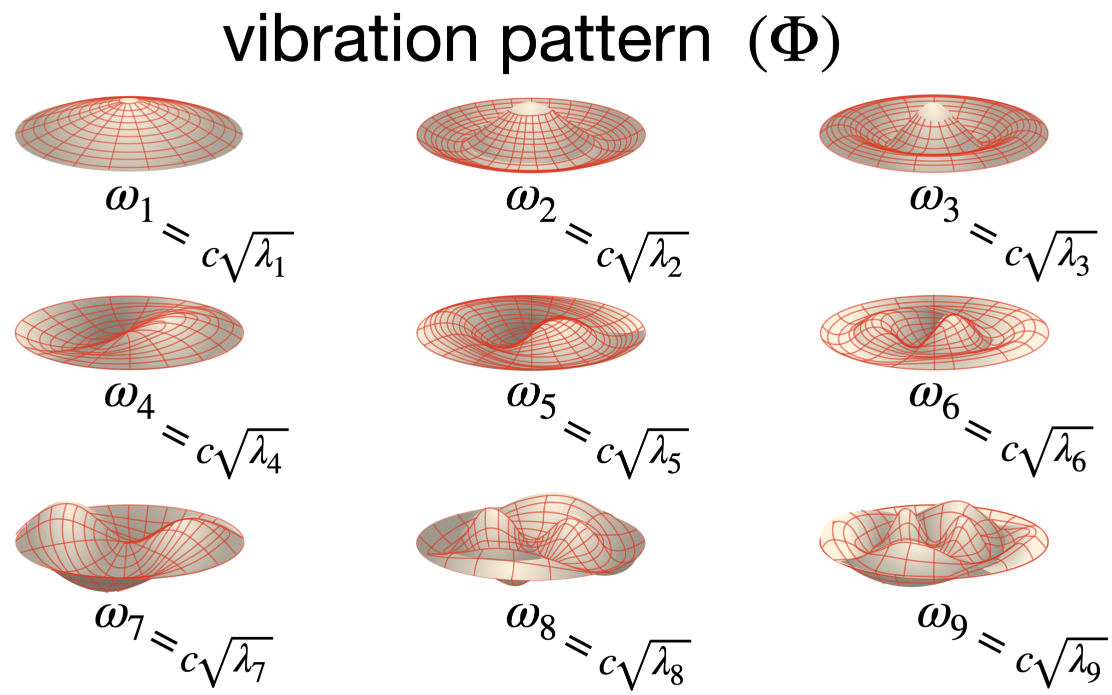


The Core of Eigenvalues & Eigenvectors

PDF link: Eigenvalues & Eigenvectors

A Little Bit of Spectral Theory

Think of the drum's vibration patterns (Φ) as the "shapes" that the membrane can take while oscillating. Each of these come with a specific frequency (determined by λ).



For the math to work, these shapes absolutely have to be *independent* of each other (to make this easier, think about perpendicular directions in geometry as an analogy). That's what *orthogonality* means in this context, that two different vibrations patterns don't "overlap" when integrated over the drum surface.

Let's see a concrete example:

The Square Drum

Let the domain be $D = [0\,,\,\pi] \times [0\,,\,\pi]$.

The PDE that we must solve now is:

$$\boxed{\Delta\Phi(x\,,\,y) + \lambda\,\Phi(x\,,\,y) = 0}$$

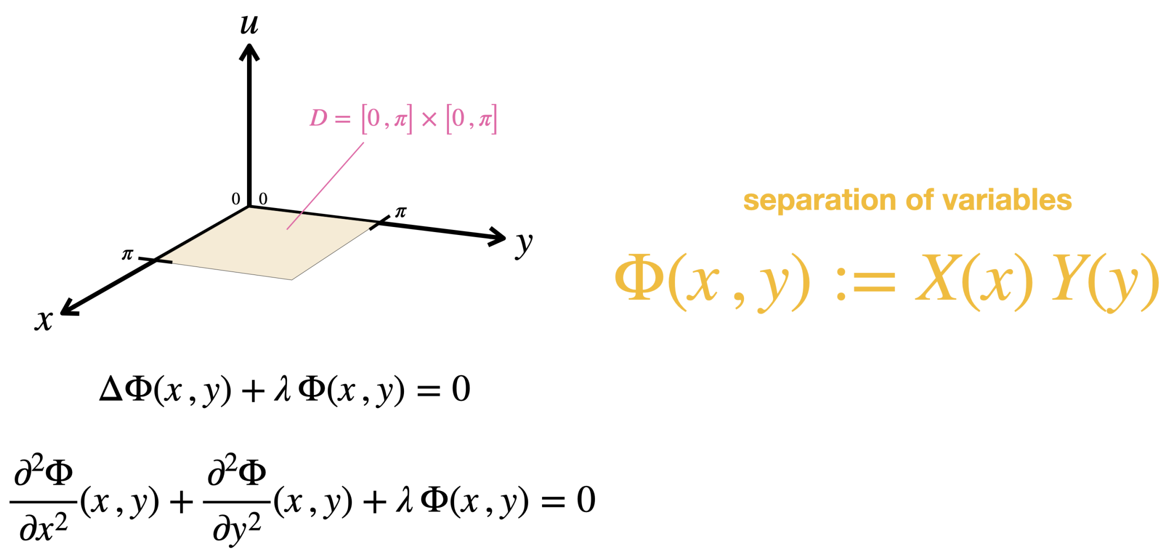
or

$$\boxed{\frac{\partial^2 \Phi}{\partial x^2}(x\,,\,y) + \frac{\partial^2 \Phi}{\partial y^2}(x\,,\,y) + \lambda\,\Phi(x\,,\,y) = 0}$$

Once again, let's use separation of variables:

$$\Phi(x\,,\,y) := X(x)\,Y(y)$$

, where $X = X(x)$ is a function that depends only on x, and $Y = Y(y)$ is a function that depends only on y.



Exercise: Try to solve the problem by finding explicit expressions for $X(x)$ and $Y(y)$, and then multiply them together to form the spatial solutions. You should use the same method we applied earlier (i.e., separation of variables). If at any point you get stuck, you can peek at a part of the solution, and then try again on your own until you can reproduce it 100% independently. Trust me, it may sound like the slow route, but this will actually accelerate your learning curve exponentially.

Solution:

Plug $\Phi(x\,,\,y) = X(x)\,Y(y)$ into $\Delta\Phi(x\,,\,y) + \lambda\,\Phi(x\,,\,y) = 0$:

$$X''(x)\,Y(y) + X(x)\,Y''(y) + \lambda\,X(x)\,Y(y) = 0$$

Dividing both sides by $X(x)\,Y(y)\ \ (\neq 0)$, give us:

$$\frac{X''(x)}{X(x)} + \frac{Y''(y)}{Y(y)} + \lambda = 0 \quad \Longrightarrow \quad \boxed{\frac{X''}{X} + \frac{Y''}{Y} = -\lambda}$$

The left-hand side is composed of the sum of a function that depends only on x $\left(\text{i.e. } \frac{X''}{X}\right)$ with another function that depedns only on y $\left(\text{i.e. } \frac{Y''}{Y}\right)$, but the right-hand side tells us that the result of this sum is constant! There is only one way in which this can be true: each function in the left-hand side must be constant ($\in \mathbb{R}$).

$$\boxed{\frac{X''(x)}{X(x)} =: -m^2} \qquad\qquad \boxed{\frac{Y''(y)}{Y(y)} =: -n^2}$$

With m, n constants, and $\lambda := m^2 + n^2$.

This is actually great news for us, because it means that we just reduced the problem from one difficult PDE to two easy (and well-known) ODEs. Do you recognize them?

$$\textbf{(1)} \quad X'' + m^2 X = 0 \quad , \qquad X(0) = X(\pi) = 0$$

$$\textbf{(2)} \quad Y'' + n^2 Y = 0 \quad , \qquad Y(0) = Y(\pi) = 0$$

The solutions are:

$$\textbf{(1)} \implies \quad X(x) = \sin(mx) \quad , \quad m = 1\,,\,2\,,\,3\,,\,...$$

$$\textbf{(2)} \implies \quad Y(y) = \sin(ny) \quad , \quad n = 1\,,\,2\,,\,3\,,\,...$$

Note: *If you don't believe me that this is the correct solution (and you shouldn't just take my word for it) check it out by plugging these solutions into the ODEs and seeing how your "machine" returns a true statement*

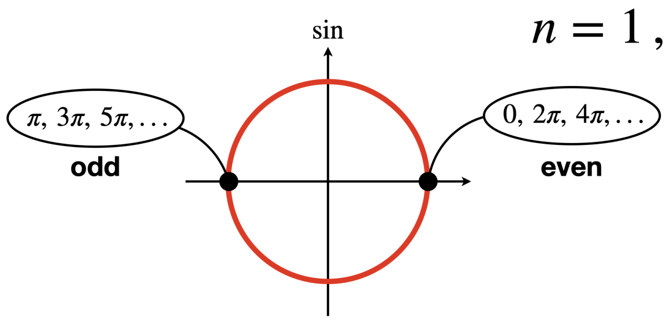
That's great! But you might be wondering: how come m and n became natural numbers? I thought the ODEs themselves admitted solutions with any positive real numbers m and n...

You're right to question that! But take a look at the boundary conditions:

$$X(0) = X(\pi) = 0 \implies \sin(m \cdot 0) = \sin(m \cdot \pi) = 0 \implies$$

$$\implies \sin(m\,\pi) = 0 \implies m = 1\,,2\,,3\,,\ldots$$

$$n = 1\,,2\,,3\,,\ldots$$



The function $\sin(mx)$ vanishes only for the cases in which $m = 1, 2, 3 \ldots$.

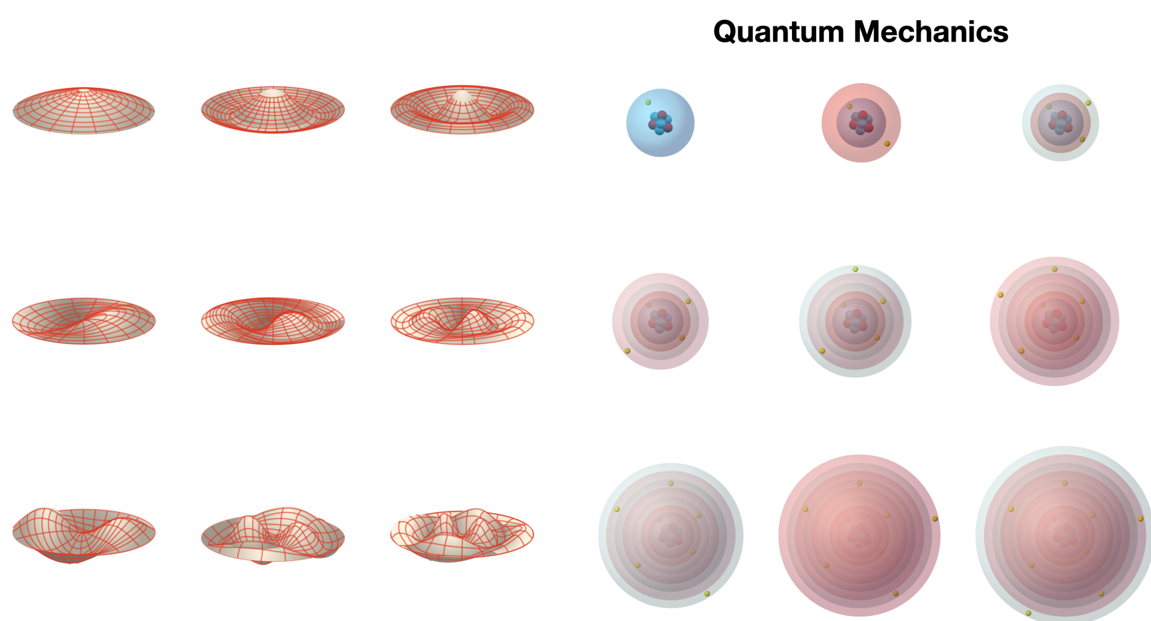
In the same way for $Y(y)$ we find out that $n = 1, 2, 3 \ldots$.

It doesn't look like much, but we just found an incredible fact about the only possible vibrations in the drum!

At first glance, you might expect the membrane to vibrate with any possible frequency, which will then form a continuous range of sounds. But the math tells us something else. Once we impose the boundary conditions (so, once the rim of the drum is fixed), then only very specific frequencies are allowed.

$$\omega_{m,n} = c\sqrt{\lambda_{m,n}} = c\sqrt{m^2+n^2} \quad , \quad m,n \in \mathbb{N}$$

The space of solutions is not continuous, it is discrete. Basically: the drum's natural vibrations are "*quantized*". This is exactly the same phenomenon that we encounter in **Quantum Mechanics**, which happens when an electron in an atom can't have any energy value, but just has discrete energy levels. In this case though instead of quantum particles, we are seeing the same mathematical principle at work but in the oscillations of a simple drum.



Therefore, the full solution is this one:

$$\boxed{u(x\,,\,y\,,\,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin(mx)\,\sin(ny)\,\left(A_{m,n}\,\cos\left(\omega_{m,n}\,t\right) + B_{m,n}\,\sin\left(\omega_{m,n}\,t\right)\right)}$$

This is the superposition of all infinitely many possible modes of vibration of the drum.

The functions $\Phi_{m,n}(x\,,\,y) = \sin(mx)\,\sin(nx)$ are the *eigenfunctions* of the problem. So, the vibration shapes (or the stationary wave patterns on the drumhead).

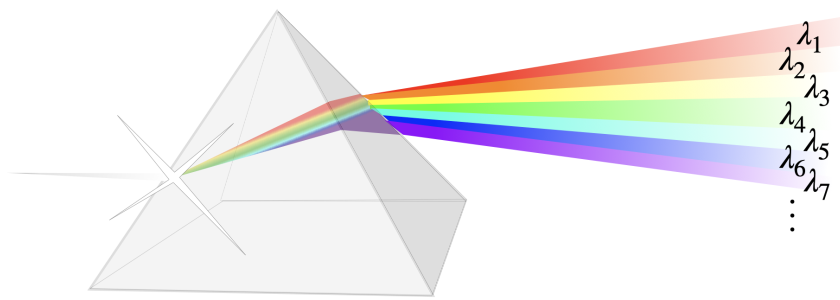
$\lambda_{m,n}$ are the *eigenvalues* of the Laplace operator Δ. These are the numbers that solve the eigenvalue equation (or, also called, *Helmholtz equation*):

$$\boxed{(\Delta + \lambda I)\;\Phi = 0} \qquad \text{with } I \text{ being the } \textbf{identity operator}.$$

$\omega_{m,n} = c\sqrt{\lambda_{m,n}}$ are the *angular frequencies*. So, the eigenvalues are not exactly the frequencies, but they do determine them.

And there is also the full infinite set of eigenvalues $\{\lambda_1\,,\,\lambda_2\,,\,\lambda_3\,,\,\dots\}$, which is called the spectrum, because (just like in physics) it lists all the "allowed frequencies" of vibration. Like a spectral decomposition of sound or light.

$$\{\lambda_1, \lambda_2, \lambda_3, \ldots\} = \text{spectrum}$$



Ok, all of this is great and beautiful, but the question remains: "**Can you hear the shape of a drum or not?!**". At this point, the conclusion might be surprising to you, because the answer is **NO! You can't!** Let me explain…

This question was actually asked all the way back in 1966, but its resolution came much later, all the way in 1992. It was answered when three mathematicians (Carolyn Gordon, David Webb & Scott Wolpert) constructed two different planar domains D_1 and D_2 that were not congruent but shared exactly the same Laplacian spectrum.

Invent. math. 110, 1–22 (1992)

Inventiones mathematicae
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Isospectral plane domains and surfaces via Riemannian orbifolds

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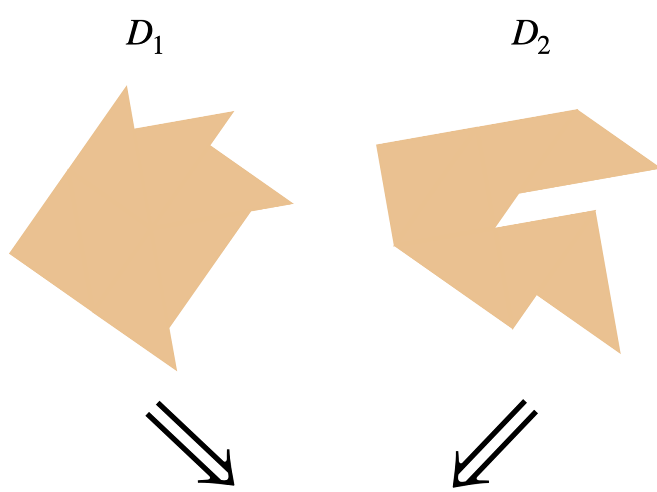
0 Introduction

Let (M, g) be a compact Riemannian manifold, perhaps with boundary. The *spectrum* of M is the sequence of eigenvalues of the Laplace-Beltrami operator $\Delta f = -\operatorname{div}(\operatorname{grad} f)$ acting on the smooth functions on M. (When $\partial M \neq \emptyset$, one can consider the Dirichlet spectrum or the Neumann spectrum.) Two Riemann-

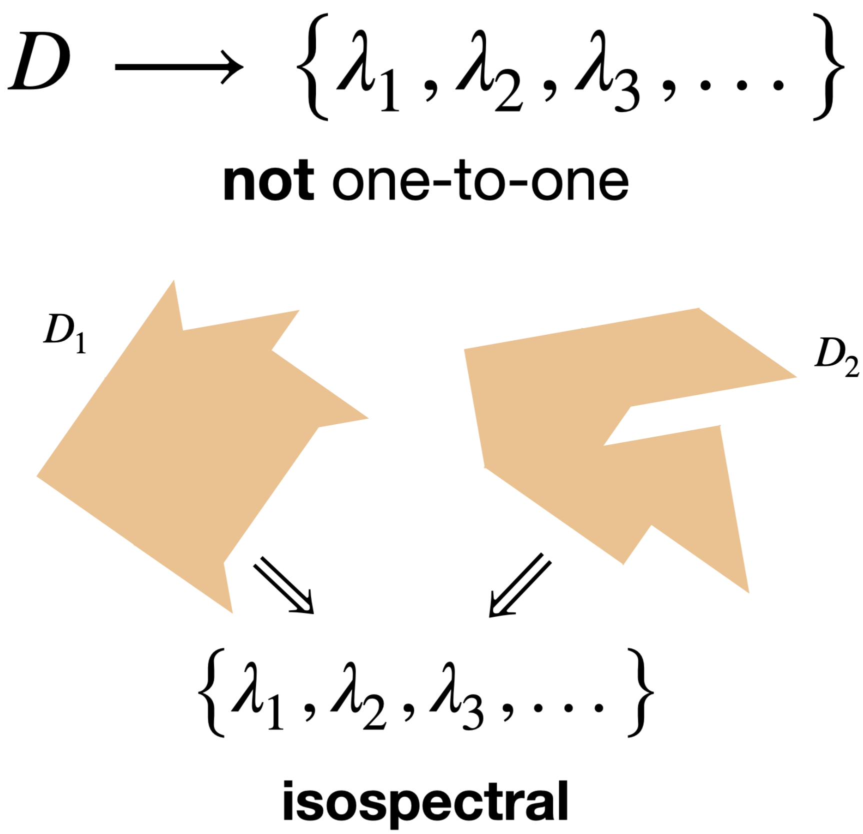
In other words, both domains had the same infinite set of eigenvalues and the same spectrum, and because of that they had the same list of natural frequencies ω. This tells us that, purely from the sound (that is, the spectrum of eigenvalues), we absolutely cannot distinguish between two different shapes of the drumhead.



$$\{\lambda_1, \lambda_2, \lambda_3, \dots\} = \text{spectrum}$$

*just a representation

Mathematically, the map from the domain to the spectrum "domain $\mapsto$ spectrum" is not *one-to-one*: different geometries can produce identical spectra. If we translate this into technical terms we can say that the two domains D_1 and D_2 are **isospectral**. And that is why it's impossible for us to reconstruct the geometry of a drumhead only from sound.



But interestingly the opposite happens to be true: given a drumhead's geometry (or the domain), we can fully determine the sound. Or more rigorously speaking, we can determine the spectrum of eigenvalues that encode the modes of vibration.

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