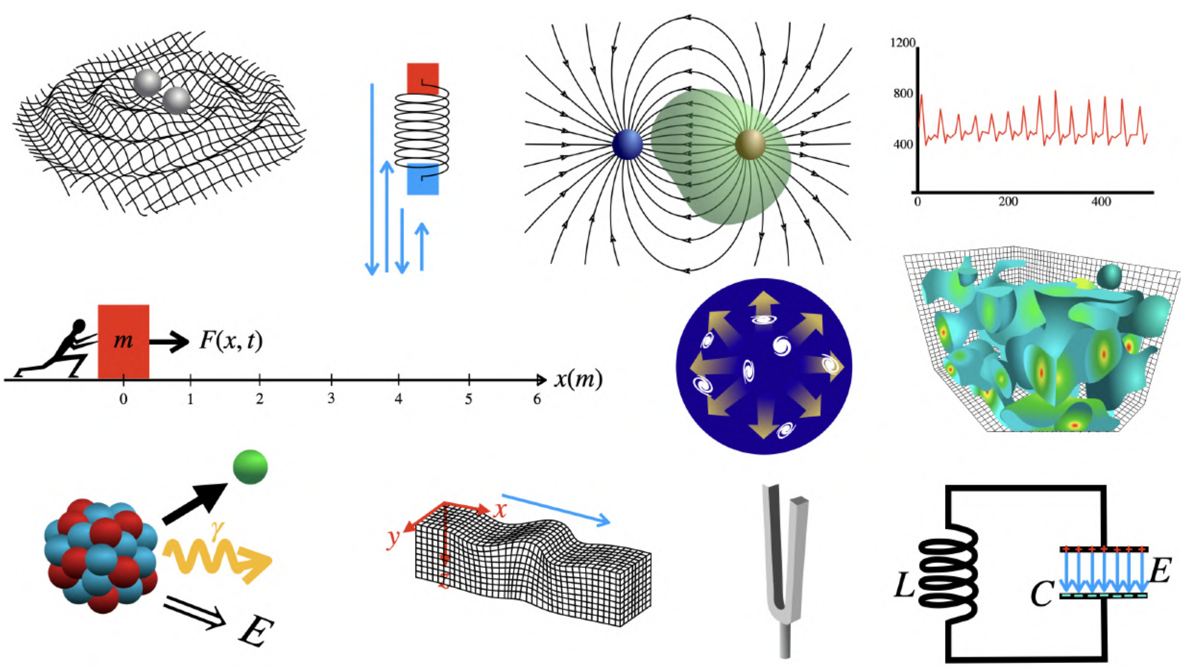
Top 25 Differential Equations of Mathematical Physics

by DiBeos



Introduction

The mathematical description of many physical systems relies on differential equations. These can be ordinary differential equations (ODEs), partial differential equations (PDEs), or even a mix of both.

The main goal in studying these equations is to find their solutions. But in practice, this is often extremely difficult — sometimes impossible. The truth is, all known methods for solving these equations depend heavily on the specific structure of each individual equation. A technique that works beautifully for one class of equations might be completely useless for another.

It's fair to say that, if we were able to solve all differential equations in mathematical physics, we would, in principle, be able to predict everything that has ever happened or will ever happen — assuming no undiscovered "exotic" physics lies beyond our current theories (which, to be honest, is totally possible).

The point, though, is that the equations that will be presented in this file are incredibly powerful. Mastering them gives you the rare ability to predict physical phenomena and even reconstruct events from the early universe, long before human consciousness ever existed.

Of course, these theories are also interesting from a purely mathematical point of view. They are full of structure, symmetry, and beauty. But beyond that, I like to think of them as glimpses into the deep truths of what we call "reality".

ODEs

An ODE of order n is an equation of the form:

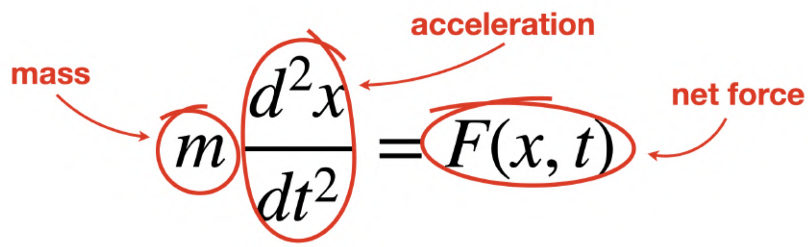
$$F(x, y, y', ..., y^{(n)}) = 0$$

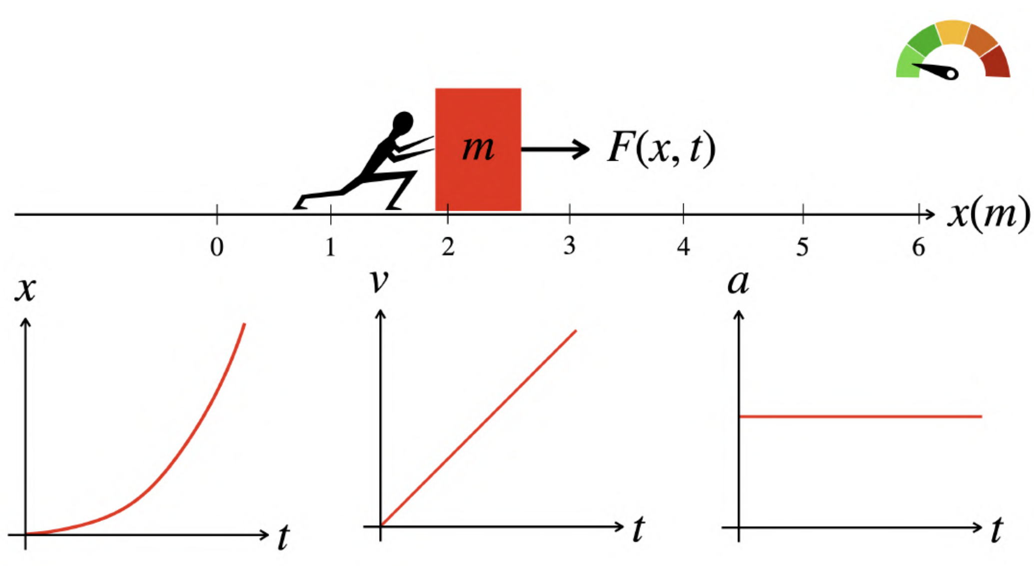
x is an *independent variable* and y is a *dependent variable*.

1 Newton's Second Law

$$m \frac{d^2x}{dt^2} = F(x, t)$$

It states that the acceleration (i.e., the rate of change of velocity over time) of a particle is proportional to the net force acting on it, scaled by its mass.

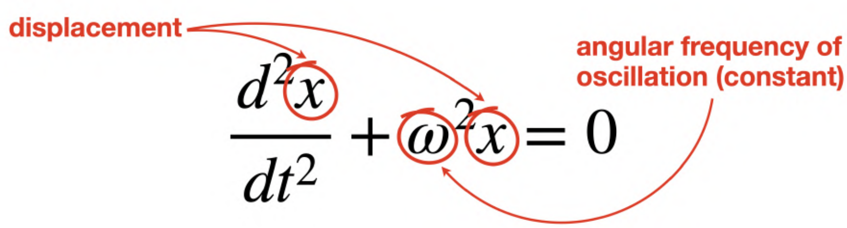


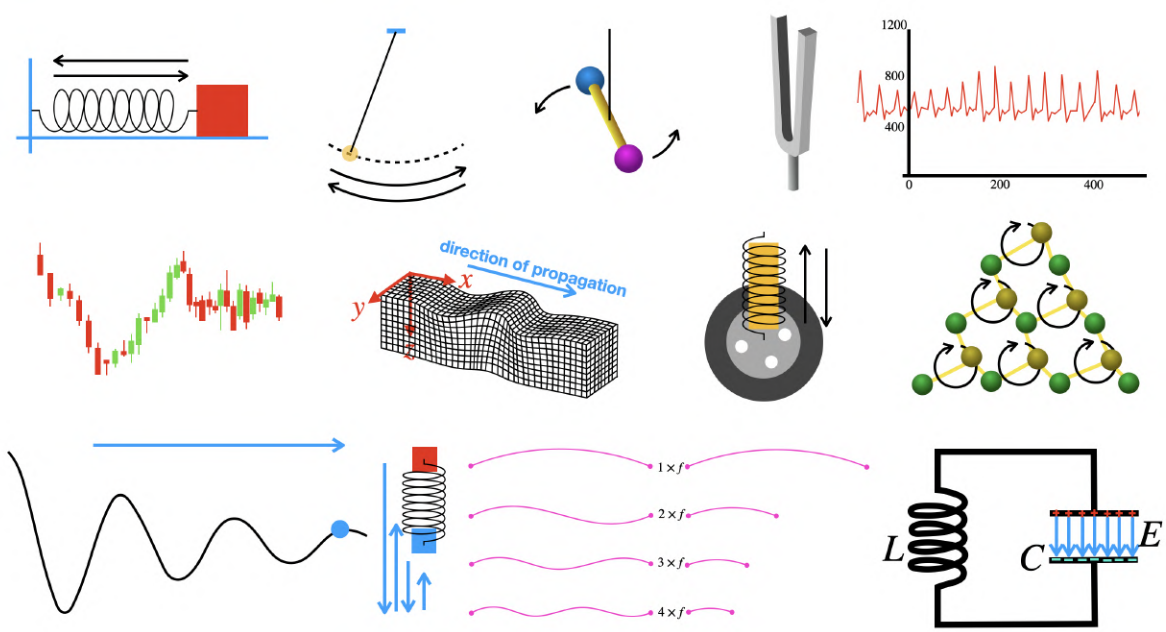


2 Simple Harmonic Oscillator

$$\frac{d^2x}{dt^2} + \omega^2 x = 0$$

This is a second order linear homogeneous equation with constant coefficients. There is a huge list of applications of this equation in Physics, Biology, Finances and Engineering.



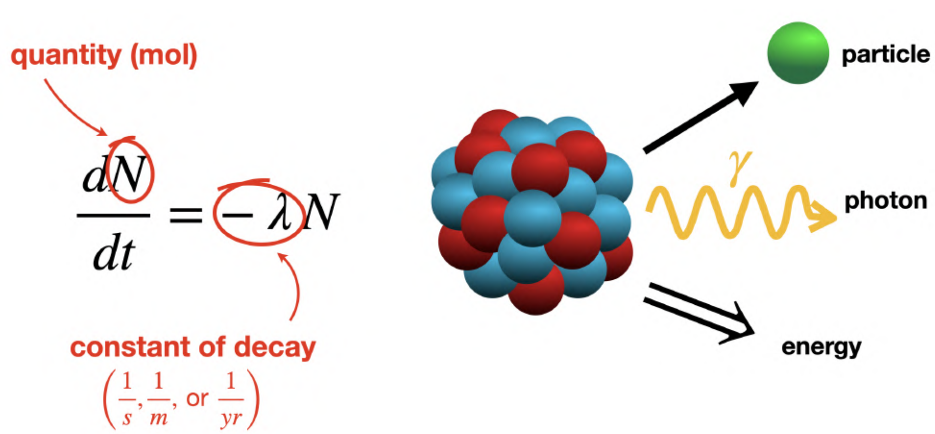


Theoretical physicists go as far as to say that: "everything in physics is either a harmonic oscillator or can be approximated to one."

3 Radioactive Decay Law

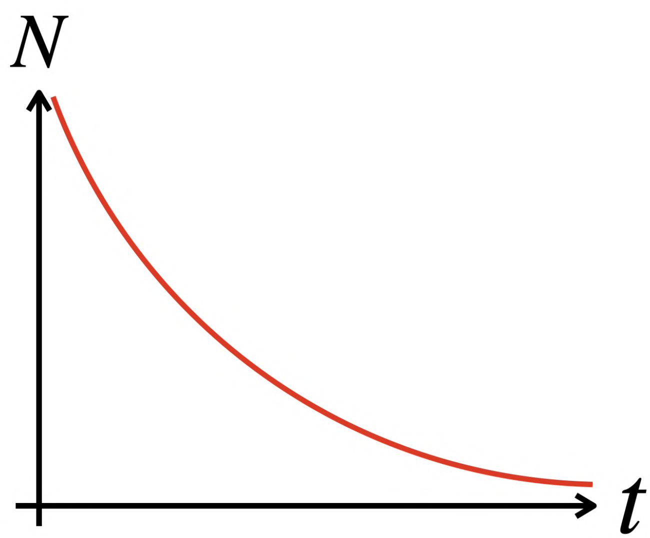
$$\frac{dN}{dt} = -\lambda N$$

$N(t)$ is the quantity of a radioactive substance, usually measured in the number of atoms or in moles. And λ is the decay constant, which determines the rate of exponential decay, and is measured in $1/s$ or $1/min$ or even $1/yr$ (since it's inverse with respect to time).



It expresses how an unstable atom decreases over time. Its solutions are exponential functions, which means that the manifestation of the force responsible for this phenomenon (called weak force) presents exponential decay:

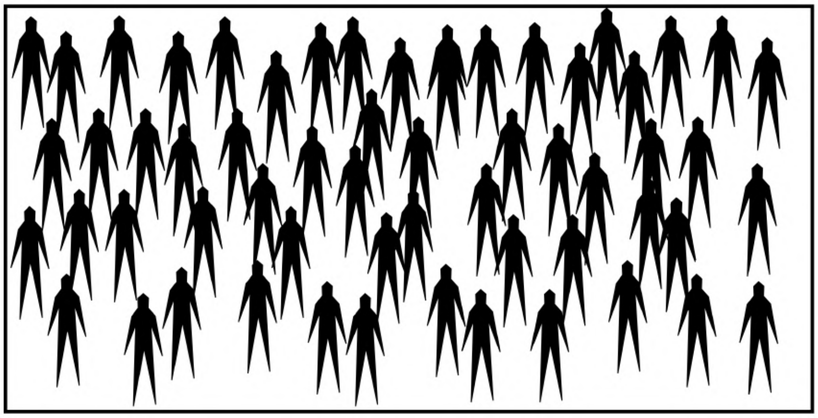
$$N(t) = N_0 e^{-\lambda t}$$



4 Logistic Growth Equation

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right)$$

Now we're not tracking the number of particles anymore, but instead the population of a closed system.



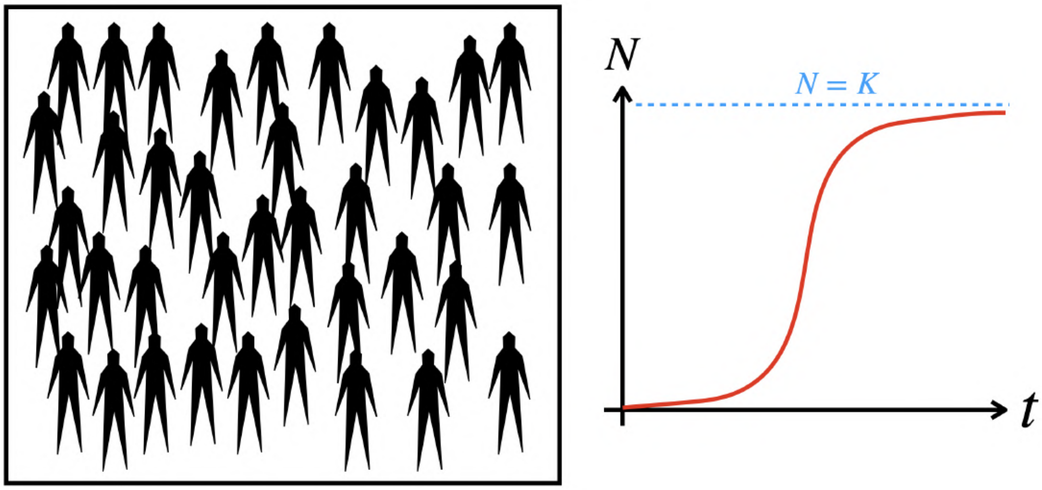
The first important thing to notice about it here is that this is the first *nonlinear* equation we've seen so far, and this is so because it includes a product of N with itself:

dependent variable

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) = rN - \frac{rN^2}{K}$$

So, we have a term with the dependent variable squared.

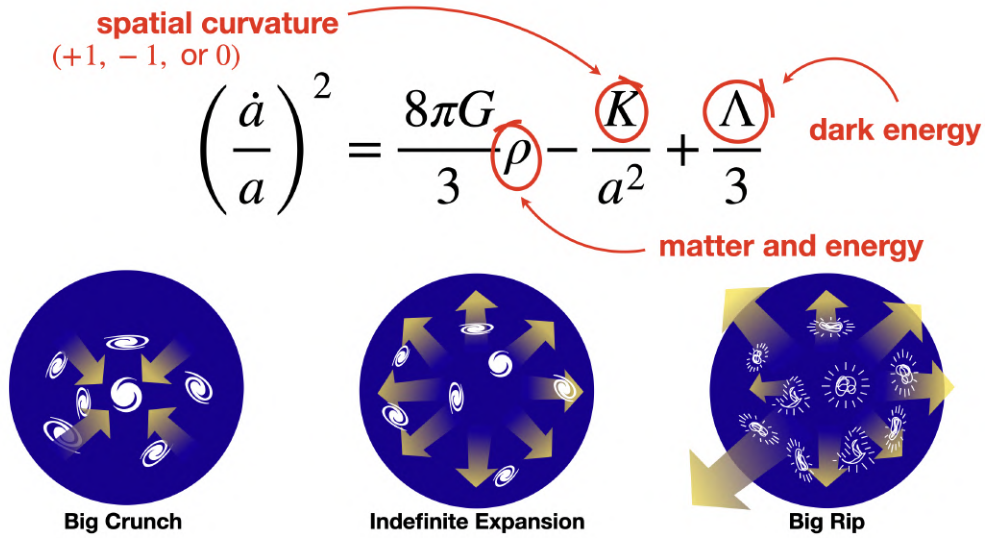
The term $\left(1 - \frac{N}{K}\right)$ produces a horizontal asymptote at $N = K$ (called the carrying capacity K).



5 Friedmann Equation

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{K}{a^2} + \frac{\Lambda}{3}$$

This innocent-looking equation describes how our universe expands or contracts over time, based on matter and energy content (ρ), spatial curvature (K) and dark energy (Λ).



$a(t)$ is the *scale factor* and its derivative $\dot{a} = \frac{da}{dt}$ is the rate of expansion.

PDEs

A PDE involves partial derivatives of a function with respect to multiple independent variables. A general PDE of order n looks like:

$$F(x_1, x_2, \ldots, x_k, u, \partial_{x_1} u, \partial_{x_2} u, \ldots, \partial^n_{x_k} u) = 0$$

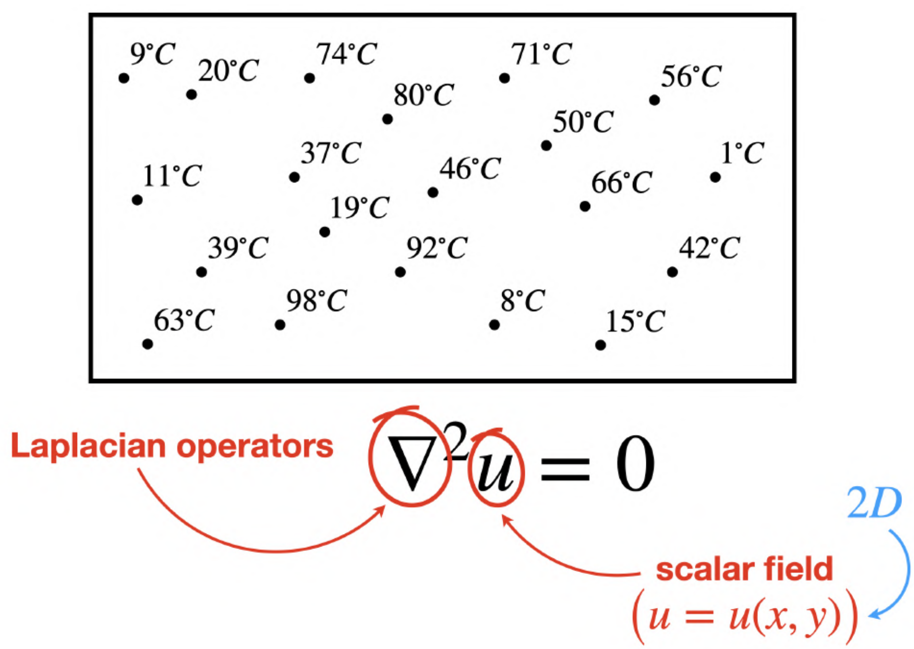
Here, $x_1, x_2, \ldots, x_k$ are the *independent variables* (such as space and time), and $u = u(x_1, \ldots, x_k)$ is the *dependent variable* (often representing a field like temperature, pressure, or wave amplitude).

Key difference with ODEs: An ODE depends on a single independent variable (like time), so it describes how something changes over time alone. A PDE depends on several variables (like space *and* time), and describes how something evolves across space and time together. For example, ODEs can model how fast a car is going, while PDEs can model how a wave spreads through a pond.

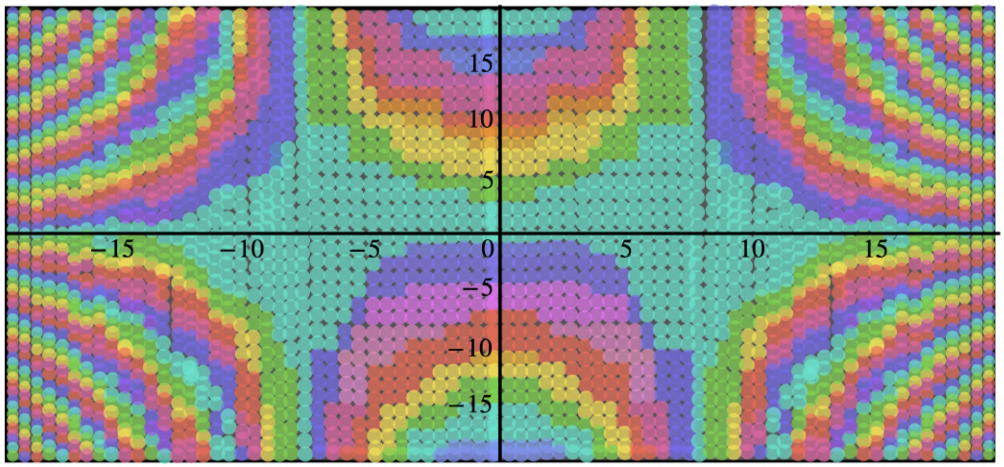
6 Laplace's Equation

$$\nabla^2 u = 0$$

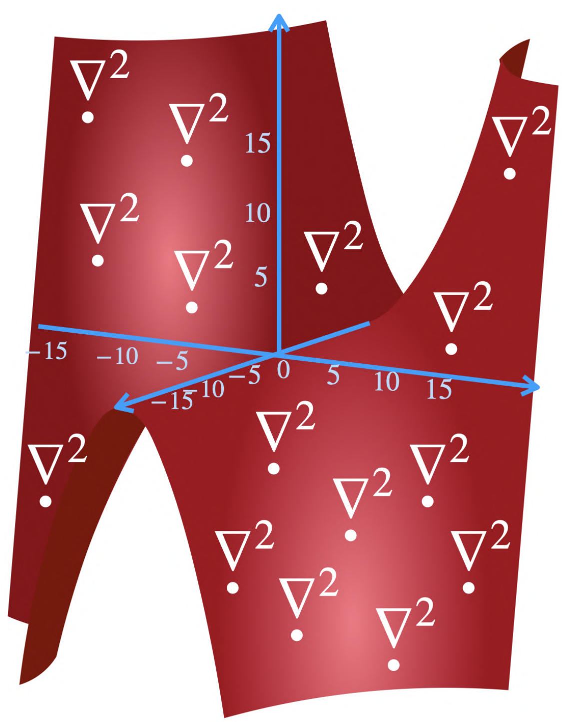
u is a scalar field. Think of a scalar field, in 2 dimensions for example, as a mapping that assigns a real (or complex) number at each point in space – like temperature, for example.



Another way to think about it is by assigning different colors to different points according to their numerical values.



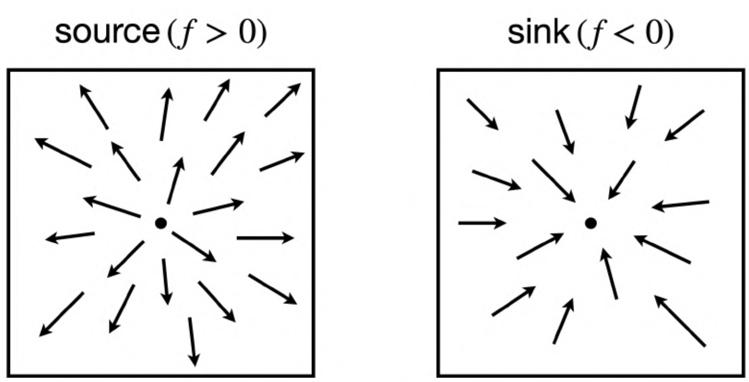
The Laplacian operator (∇^2) measures how much the function at a point differs from its surrounding values, i.e. is this point a *peak*, a *valley*, or just *flat* compared to its neighbors?



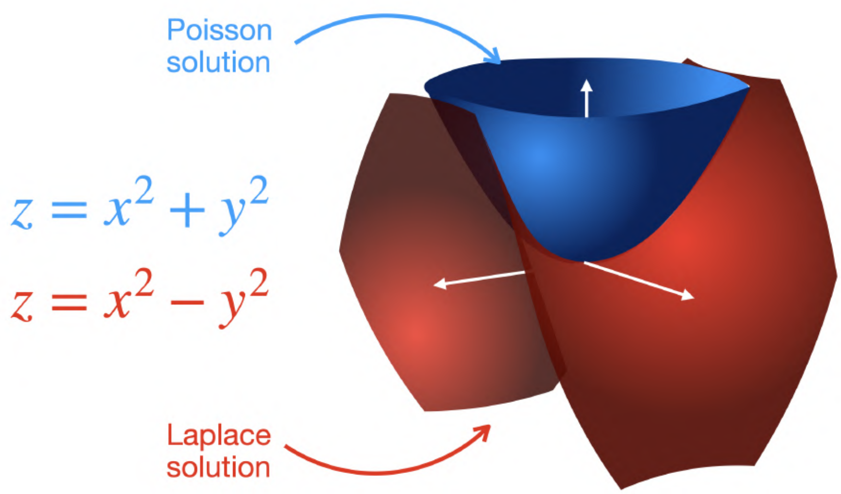
7 Poisson's Equation

$$\nabla^2 u = f(x, y)$$

The only difference between Poisson's and Laplace's equations is the term $f(x, y)$. This function (f) is a *source* if it's *positive*, and a *sink* if it's *negative*.



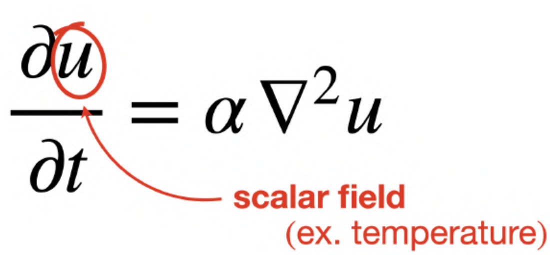
Since the right-hand side (RHS) of Laplace's equation is zero, there isn't such behavior. Solutions to Laplace's equation are commonly called *harmonic functions*, and can model, for example, a region in space with electrostatic potential, but no charges. Meanwhile, Poisson's solutions would include charges in it.

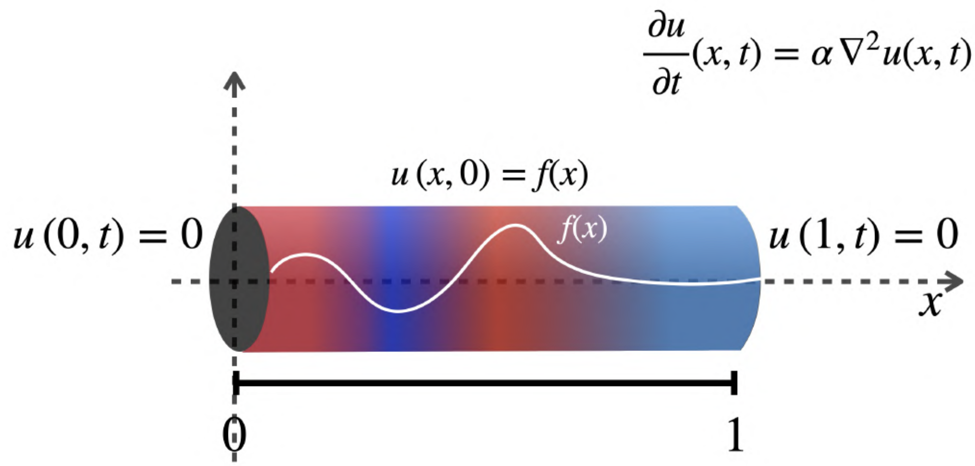


8 Heat (diffusion) Equation

$$\frac{\partial u}{\partial t} = \alpha \nabla^2 u$$

It describes how quantities, given by the scalar field u, spread out over time.





But it's not limited to physics. It also describes the diffusion of nutrients in tissues (like oxygen in capillaries), or even the prices of financial derivatives in the stock market – most notably through the famous:

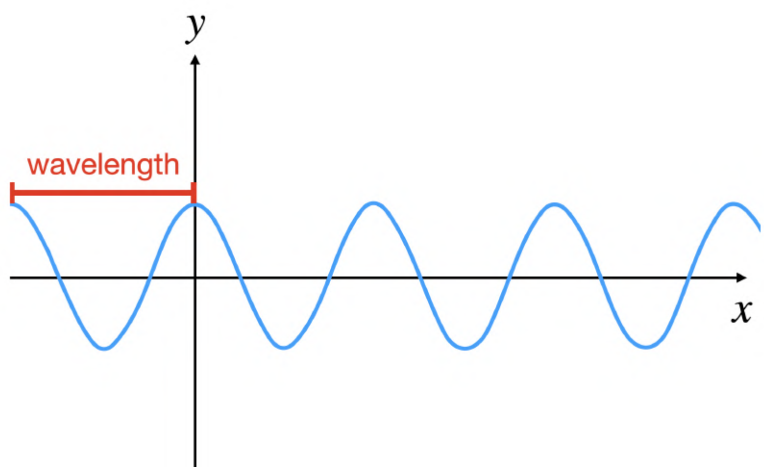
9 Black-Scholes equation

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0$$

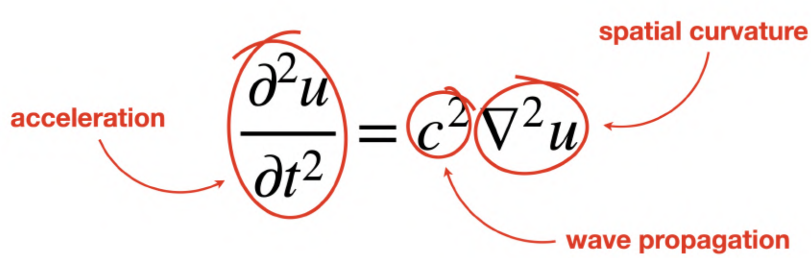
It is mathematically equivalent to the diffusion equation under change of variables. This equation helped build Wall Street empires and made many quant traders multi-millionaires.

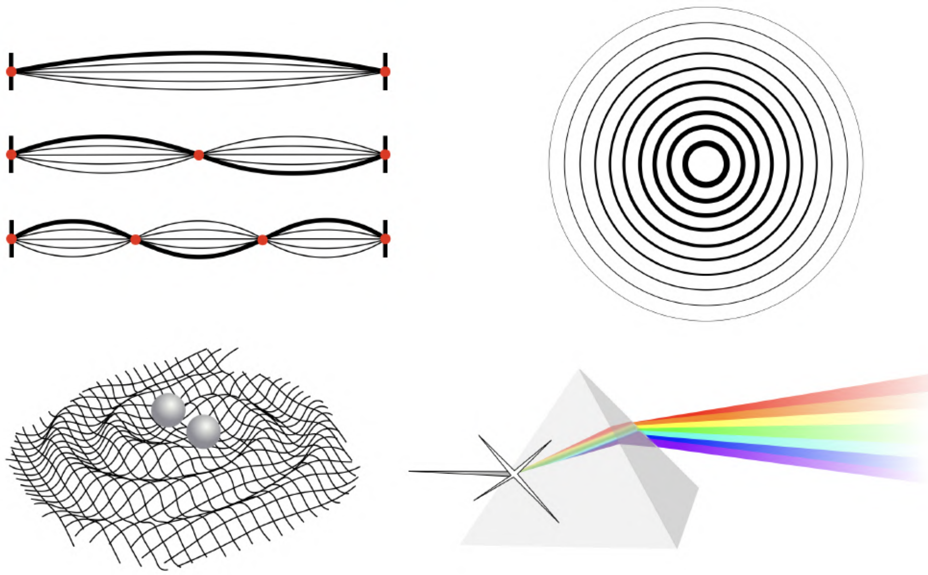
10 Wave Equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \nabla^2 u$$



It says that the acceleration of a field u (so, its second derivative) at a point is proportional to the spatial curvature. It can describe vibrations of strings, sound waves, light waves, gravitational waves, and so on.





11 Schrödinger Equation (Time-Dependent)

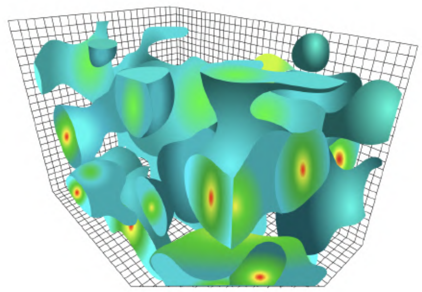
$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$

This equation tells us how a quantum state evolves in time. The wave-function $\psi = \psi(x, t)$ contains all the possible pieces of information that are measurable about a particle or a system of particles.

wavefunction $\psi(x,t)$

Hamilton operator (extract energy info)

$$i\hbar\frac{\partial\psi}{\partial t} = \hat{H}\psi$$



The *Hamiltonian operator* is responsible for extracting information about the particle's total energy (both kinetic and potential).

Maxwell's Equations

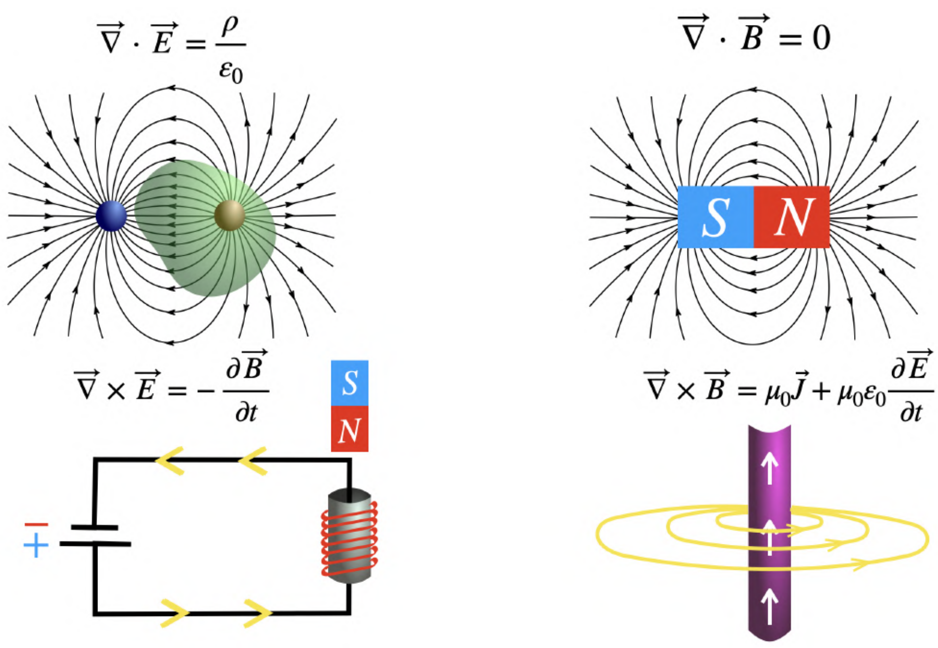
12) Gauss's Law for Electricity $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$

13) Gauss's Law for Magnetism $\vec{\nabla}\cdot\vec{B} = 0$

14) Faraday's Law of Induction $\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$

15) Ampère-Maxwell Law $\vec{\nabla}\times\vec{B} = \mu_0\vec{J} + \mu_0\varepsilon_0\frac{\partial\vec{E}}{\partial t}$

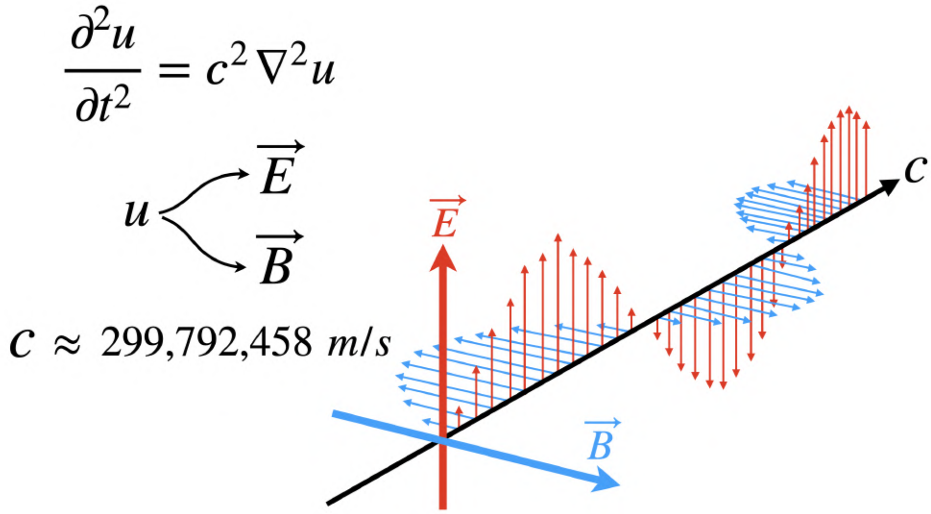
These are obviously more than one, indeed 4 PDEs. And they are put together because they explain all classical electromagnetism phenomena in nature.



They are called Maxwell's equations not just because he was the one who complied them together, but because he discovered something extraordinary, using mathematics alone:

If you combine them in a certain way, you get a wave equation, and after calculating the speed of this wave, the numerical value matches almost perfectly the speed of light in vacuum. The conclusion is:

"Light is an electromagnetic wave."



16 Klein-Gordon Equation

$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

Now we are in the realm of Relativistic Quantum Mechanics.
Do you see this little square right here? Cute, huh?! It's called the *d'Alembert operator*, and it means this expression:

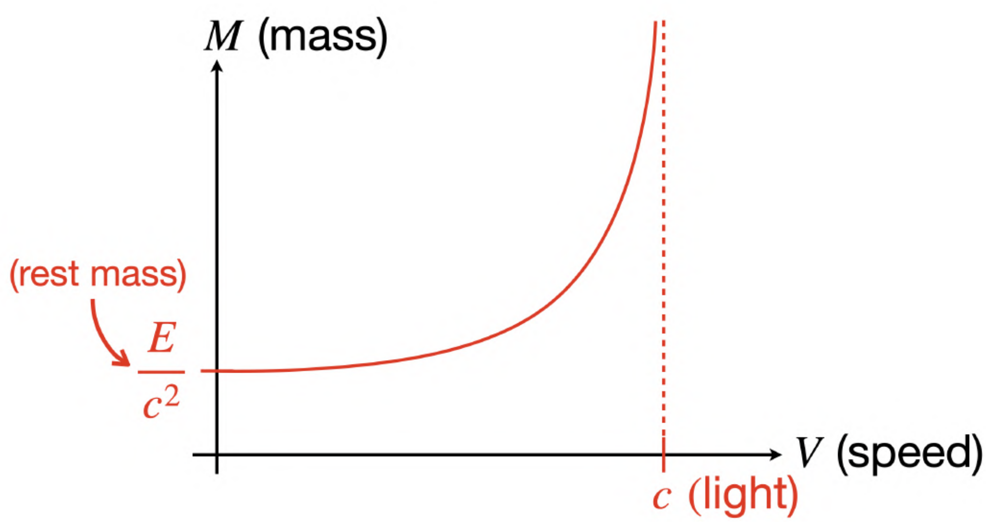
$$\left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

d'Alembert operator $= \frac{\partial^2}{\partial t^2} - c^2\nabla^2$

So, the explicit version of this equation is this:

$$\frac{\partial^2\psi}{\partial t^2} - c^2\,\nabla^2\psi + \frac{m^2c^2}{\hbar^2}\psi = 0$$

The Klein-Gordon equation is a generalization of our good old wave equation, but this time we include relativistic mass effects.

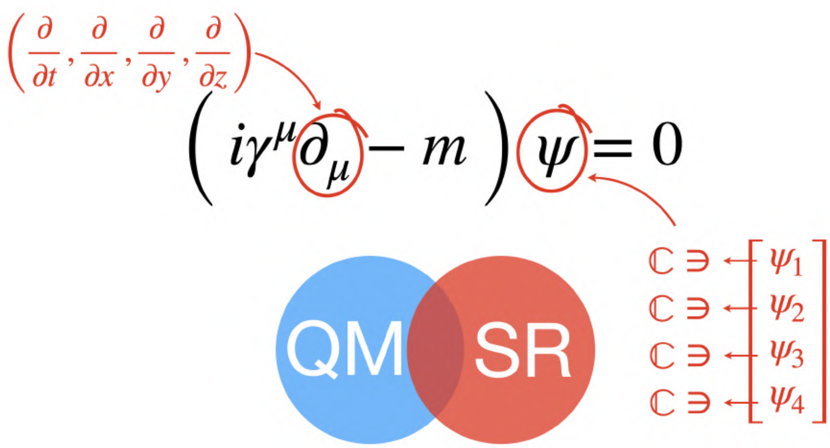


In other words, this equation takes into account the physical fact that a particle's mass can't be treated as a fixed constant as it approaches the speed of light. If the particle were to reach the speed of light, its relativistic mass would asymptotically increase towards infinity.

17 Dirac Equation

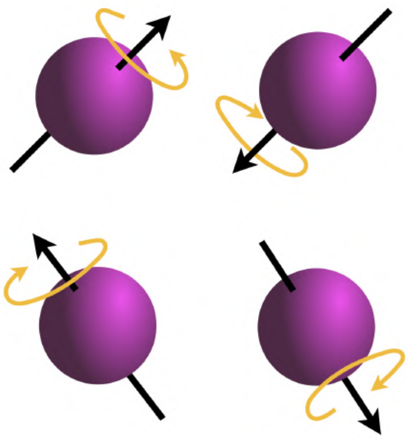
$$\left(i\gamma^\mu \partial_\mu - m \right) \psi = 0$$

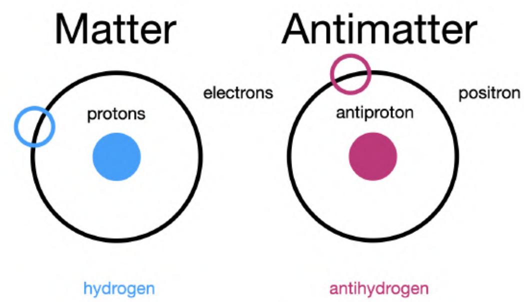
This that you are looking at right now is the very equation that unifies Quantum Mechanics and Special Relativity!



You will also find it written on Paul Dirac's tomb, in the Westminster Abbey, in London, near the tomb of Sir Isaac Newton. And, by the way, I (Luca) was fortunate enough to visit the site in person!

Anyway, this equation also introduces the concept of the *spin* of a particle in a very natural way, and it predicts *antimatter*.





ψ here is a 4-component vector such that each component is complex. So, this field contains 8 degrees of freedom.

$$\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{bmatrix} = \begin{bmatrix} Re\left(\psi_1\right) + i\,Im\left(\psi_1\right) \\ Re\left(\psi_2\right) + i\,Im\left(\psi_2\right) \\ Re\left(\psi_3\right) + i\,Im\left(\psi_3\right) \\ Re\left(\psi_4\right) + i\,Im\left(\psi_4\right) \end{bmatrix}$$

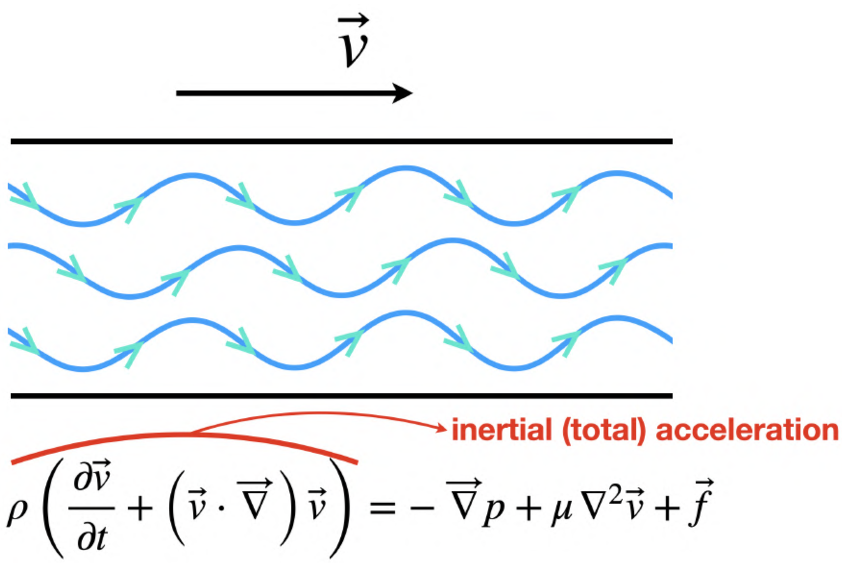
8 degrees of freedom

18 Navier-Stokes Equation

$$\rho\left(\frac{\partial \vec{v}}{\partial t} + \left(\vec{v}\cdot\vec{\nabla}\right)\vec{v}\right) = -\,\vec{\nabla}p + \mu\nabla^2\vec{v} + \vec{f}$$

$$\vec{\nabla}\cdot\vec{v} = 0$$

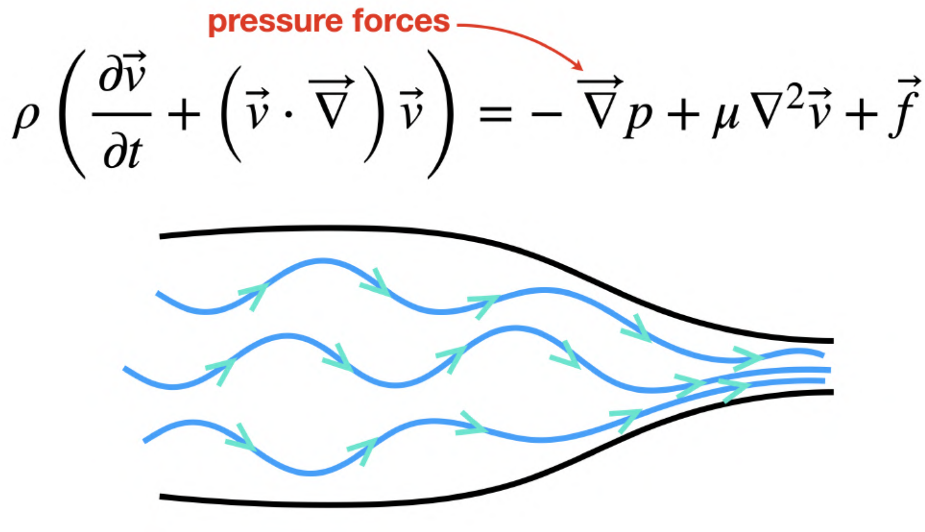
These, right here, are basically Newton's second law applied to fluid elements.



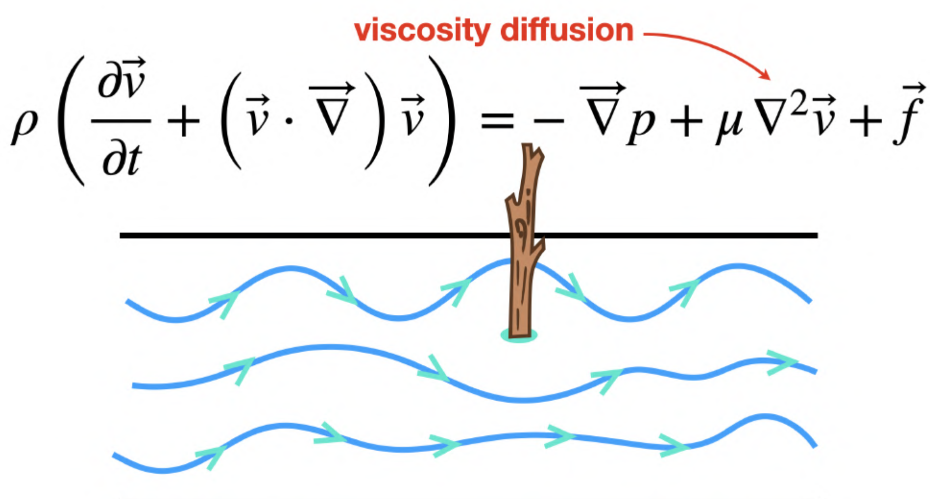
The left-hand side (LHS) represents *inertial acceleration*, i.e. the total acceleration experienced by a fluid element.

The RHS represents

Pressure forces:



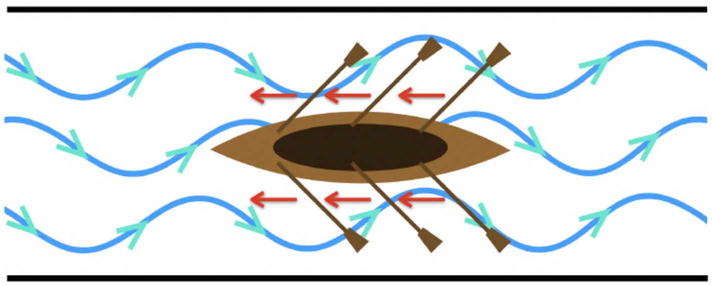
Viscosity diffusion:



External forces:

external forces

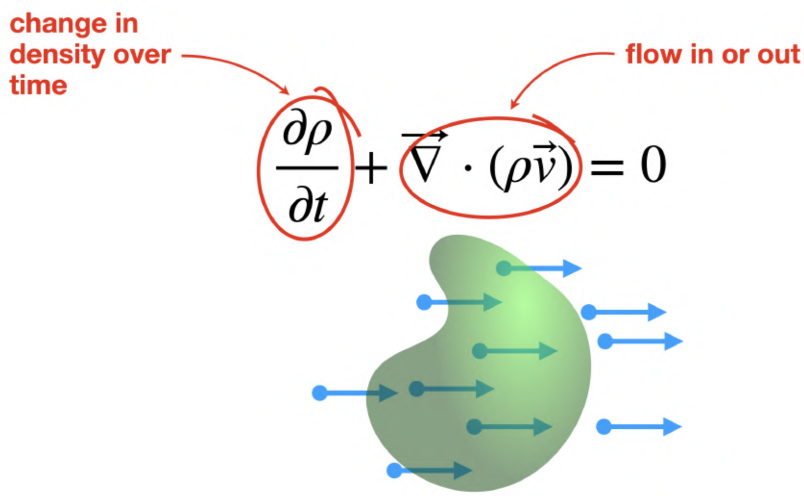
$$\rho\left(\frac{\partial\vec{v}}{\partial t}+\left(\vec{v}\cdot\overrightarrow{\nabla}\right)\vec{v}\right)=-\overrightarrow{\nabla}p+\mu\nabla^2\vec{v}+\vec{f}$$



19 Continuity Equation

$$\frac{\partial\rho}{\partial t}+\overrightarrow{\nabla}\cdot(\rho\vec{v})=0$$

This equation has many, many, different applications, but in physics it's often used to express the fact that mass can't just appear or disappear. Any change in density over time $\left(\frac{\partial\rho}{\partial t}\right)$ in a region must be due to flow into or out of that region.



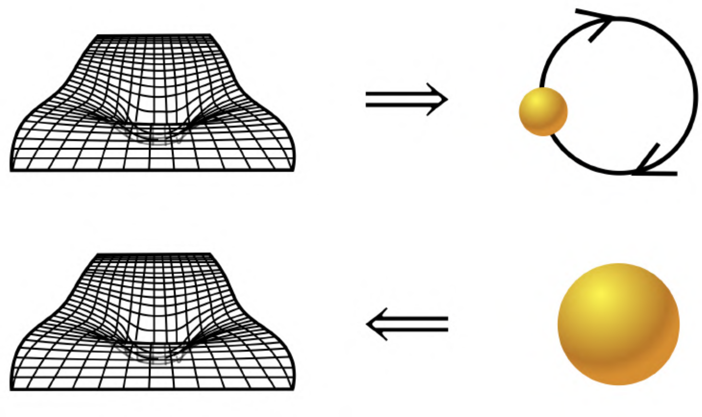
An informal way of describing what this equation tells us is:

"Everything that comes in is equal to what comes out plus what stayed in."

20 Einstein Field Equations

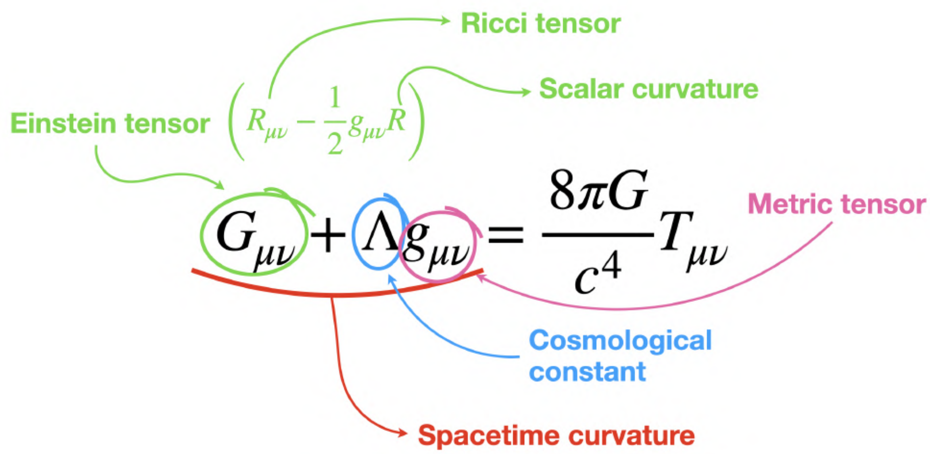
$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

This set of equations is physically interpreted as: "Spacetime tells matter how to move, and matter tells spacetime how to curve."

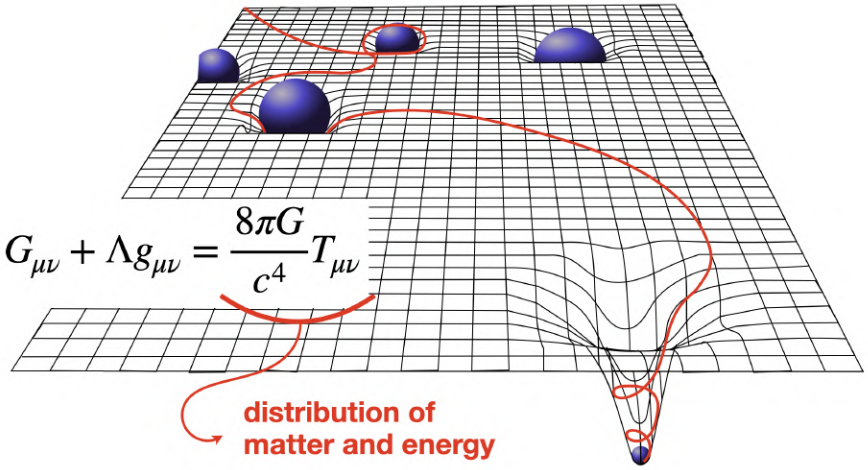


What creates what? Hard to say…

The LHS describes how spacetime curves, with Einstein tensor (which combines the Ricci tensor and scalar curvature) and the cosmological constant (responsible for the accelerated expansion of the universe) multiplied by the metric tensor.



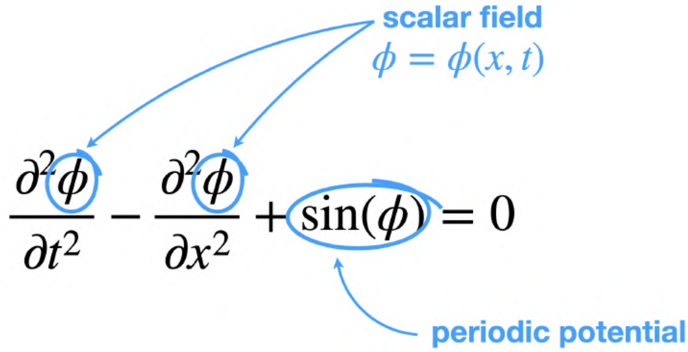
The RHS, instead, describes the distribution of matter and energy that causes the curvature.

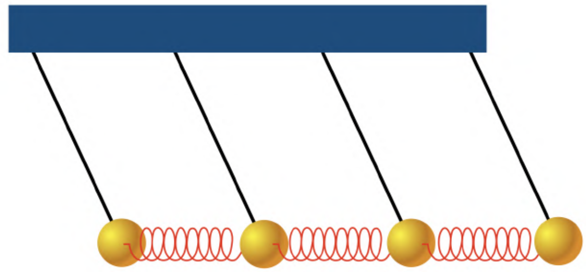


21 Sine-Gordon Equation

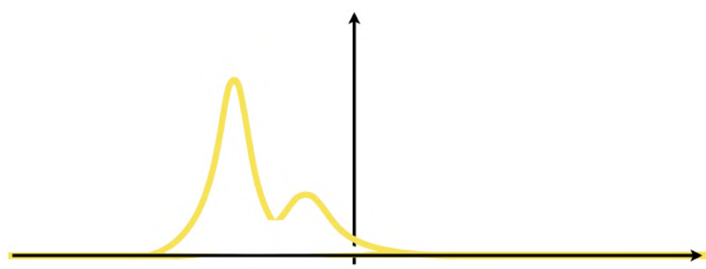
$$\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \sin(\phi) = 0$$

ϕ is a scalar field. This is a nonlinear wave equation with periodic potential (which is the sine term). It's used to model waves in a periodic medium, like a chain of pendulums attached by strings.





One type of solution is called a *soliton*. This is a sort of pulse, but with the extra condition that it's stable and presents localized wave packets that behave like particles and pass through each other without changing shape.

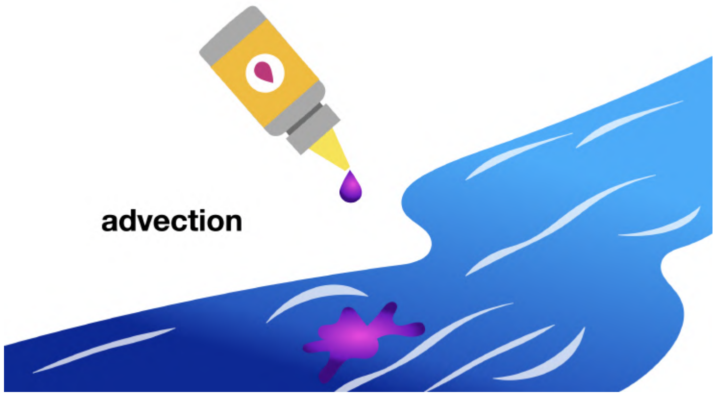


22 Burger's Equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}$$

This equation models a process called *advection*, which is a type of non-linear transport.

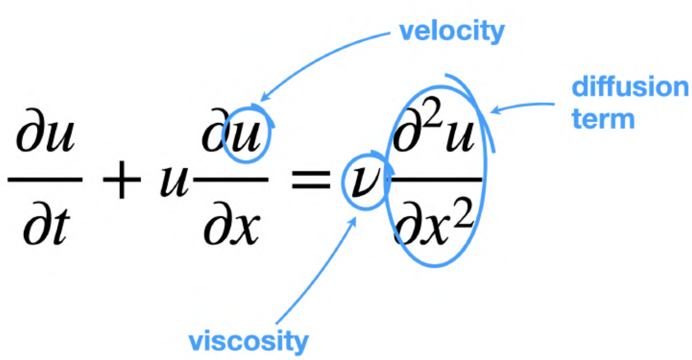
Imagine dropping dye in a flowing river. As the water moves, the dye is carried along with the current. That transport is advection.



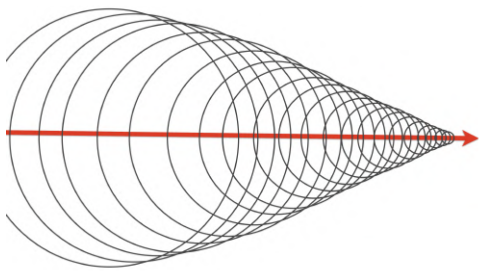
The scalar field u is usually velocity.

ν is the viscosity of the fluid.

Notice how the RHS is similar to the heat equation. It's called the diffusion term $\nu \frac{\partial^2 u}{\partial x^2}$.



When a shock wave forms, the profile of the scalar field u tends, over time, to develop a sharp corner. This steep jump can be smoothed out by increasing the viscosity of the medium.



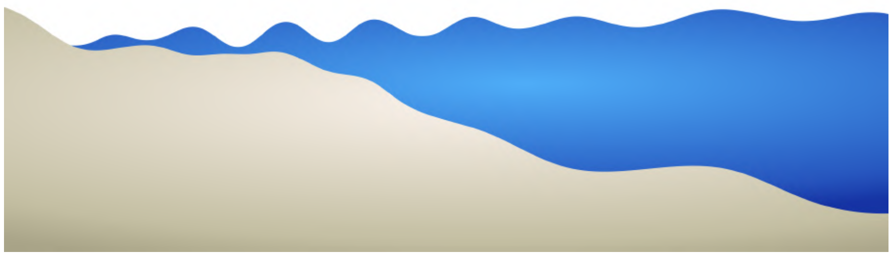
Think of viscosity as internal friction at the molecular level. It resists abrupt changes in velocity.

23 Korteweg-de Vries (KdV) Equation

$$\frac{\partial u}{\partial t} + 6u\,\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

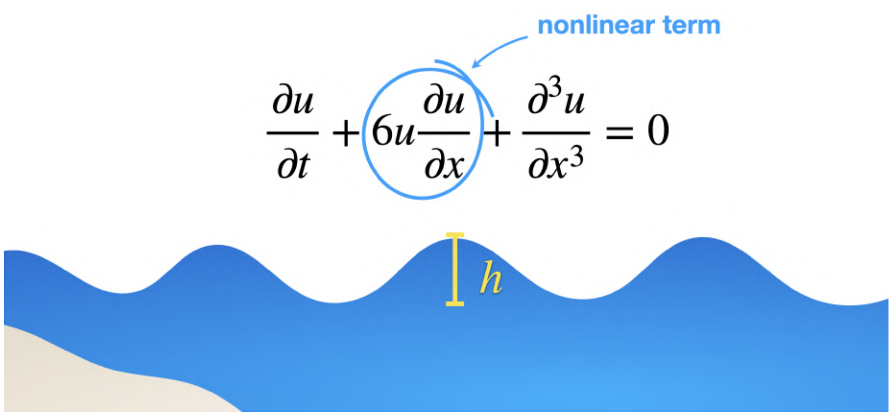
This equation models shallow water waves (those that do not “break”, yet), with long wavelengths.

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

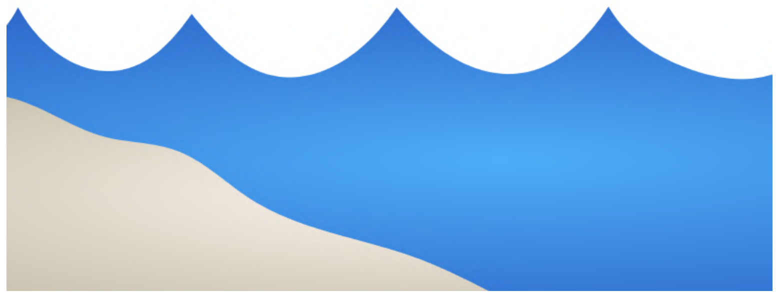


The function $u = u(x, t)$ usually tracks the wave profile, i.e. the height of the wave.

$$\frac{\partial u}{\partial t} + 6u\frac{\partial u}{\partial x} + \frac{\partial^3 u}{\partial x^3} = 0$$

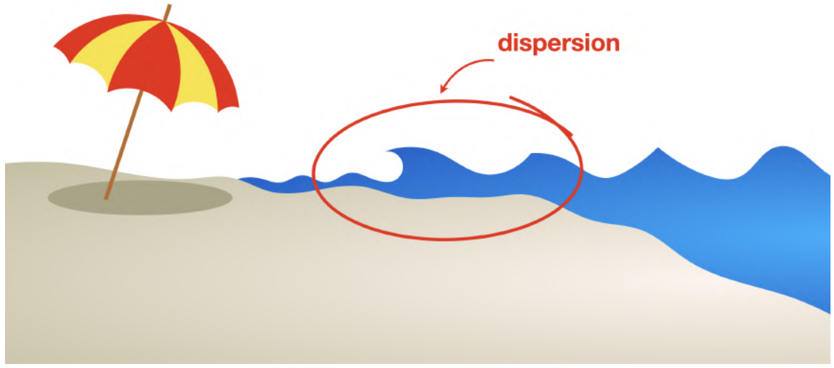


It has a nonlinear term $6u\frac{\partial u}{\partial x}$, since the function u is multiplied by its own derivative with respect to horizontal position x. This nonlinearity can be interpreted with the fact that the evolution of the wave depends on its own amplitude. The higher the amplitude, the faster it changes, and as a consequence the waves get more and more distorted.



In a linear wave, in contrast, all parts of the wave travel at the same speed. But in a nonlinear wave (like with the KdV equation right here), the peak travels faster than the points in the bottom, because the speed depends on the amplitude.

This causes the wavefront to tilt forward, so it becomes steeper over time, like a water wave approaching the shore before it breaks. What happens, then, is a process called "dispersion", which just means that different wavelengths travel at different speeds. That's what the term with the third derivative ... does. So, instead of steepening, the wave wants to spread out and flatten.



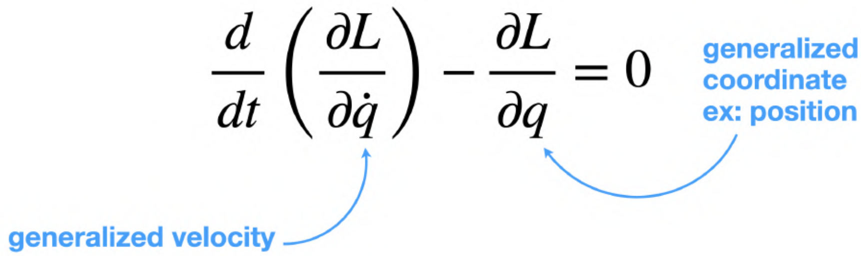
Try to notice this effect next time you go to the beach…

24 Euler-Lagrange Equation

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0$$

It tells us how a system evolves in time by finding the path that minimizes the *action*. This is called the “Hamilton principle”. Instead of using forces (like in Newton’s equation), it uses energies.

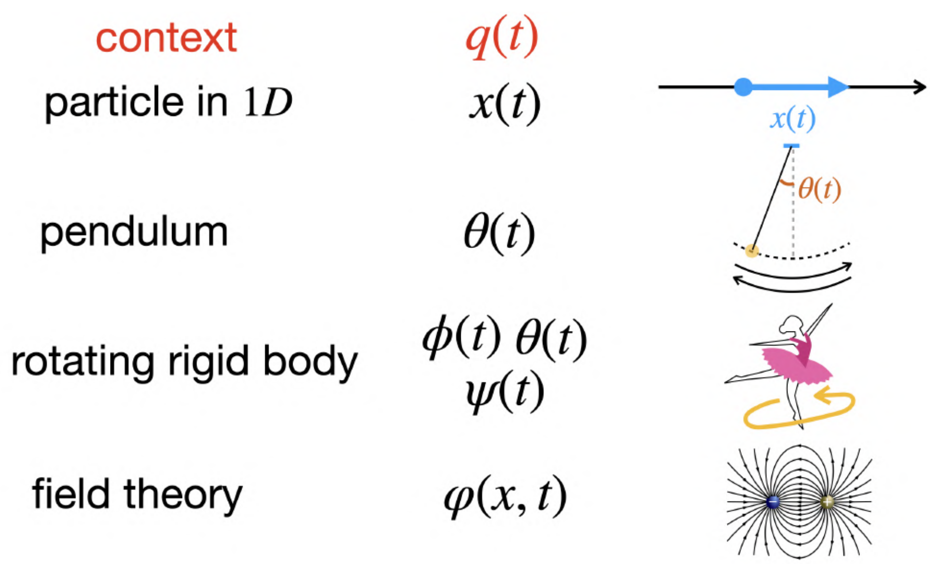
This equation assumes that the system is *conservative*, which means that the total energy is conserved. It also assumes that the evolution follows the *principle of least action*.



$q(t)$ is the "generalized coordinate" and $\dot{q}(t)$ is the "generalized velocity". The word "generalized" here means that it's not limited to a specific coordinate system, like Cartesian coordinates for example.

It really depends on the context:

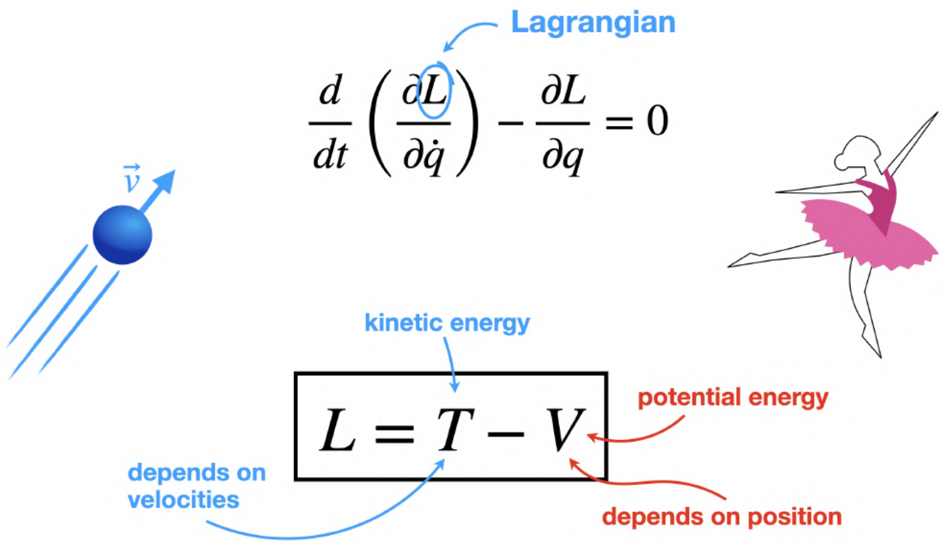
For modelling a particle in $1D$, $q(t)$ is simply its position $x(t)$. For a pendulum, $q(t)$ might be the angle $\theta(t)$ from the vertical axes. In a rotating rigid body, $q(t)$ can track any of the "Euler angles": $\phi(t)$, $\theta(t)$, $\psi(t)$. And so on...



For modeling a particle in $1D$, $\dot{q}(t)$ is its velocity $\dot{x}(t)$. For a pendulum, the angular velocity $\dot{\theta}(t)$. And so on...

Back to the Euler-Lagrange equation, the function $L = L(q, \dot{q}, t)$ is called the *Lagrangian*.

In classical mechanics of particles and rigid bodies, the Lagrangian is the difference between the *kinetic* and the *potential* energies:



In more complex systems, though, the Lagrangian is defined as the mathematical object that encodes the dynamics of the process. For instance:

In Electromagnetism (EM) and Quantum Field Theory (QFT) we work with the Lagrangian density $\mathcal{L}$ instead, which is a real-valued function of fields and their derivatives. It's analogous to the Lagrangian in classical mechanics, but instead of dealing with a finite number of particles, it describes fields defined over all space and time.

Lagrangian Density : $\mathscr{L} = \mathscr{L}\left(\phi_a;\, \partial_\mu \phi_a\right) =$ **real valued function**

The Euler-Lagrange equation, then, for fields, take this form:

$$\frac{\partial \mathscr{L}}{\partial \phi} - \partial_\mu \left(\frac{\partial \mathscr{L}}{\partial(\partial_\mu \phi)} \right) = 0$$

field theory

In classical EM, the Lagrangian density is:

$$\mathscr{L} = \frac{1}{4} F_{\mu\nu} F^{\mu\nu}$$

electromagnetic field strength tensor

Here, $F_{\mu\nu}$ is the "electromagnetic field strength tensor". And this Lagrange density, when inserted into the Euler-Lagrange equations for fields, naturally gives us the 4 Maxwell's equations!

$$\mathcal{L} = \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \qquad \frac{\partial \mathcal{L}}{\partial \phi} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\right) = 0 \rightarrow$$

$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

In QFT, the Lagrangian density $\mathcal{L}$ varies depending on the field and on the interactions in it. For example, for a *free Klein-Gordon scalar field*, this is the form of the Lagrangian density:

$$\boxed{\mathcal{L} = \frac{1}{2}\partial^\mu \phi \, \partial_\mu \phi - \frac{1}{2}m^2\phi^2}$$

free Klein Gordon scalar field

This Lagrangian density, when inserted into the Euler-Lagrange equations for fields, naturally gives us the Klein-Gordon equation!

$$\mathscr{L} = \frac{1}{2}\partial^\mu\phi\,\partial_\mu\phi - \frac{1}{2}m^2\phi^2 \qquad \frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)}\right) = 0 \;\longrightarrow\; \left(\Box + \frac{m^2c^2}{\hbar^2}\right)\psi = 0$$

25 Hamilton-Jacobi Equation

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

This is a description of the same classical mechanics we've seen earlier, but through a different pair of lenses.

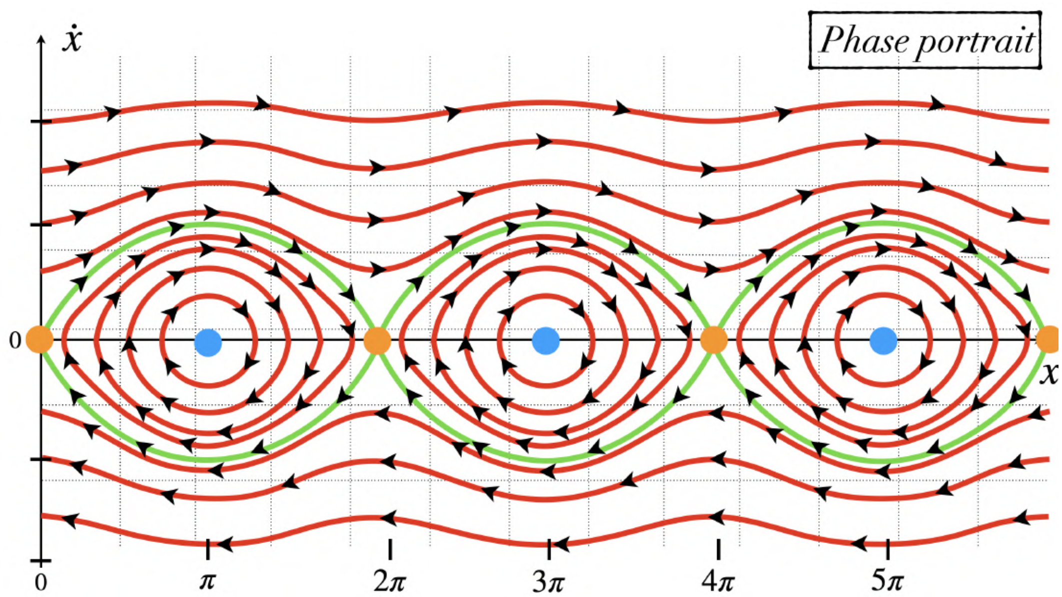
action

$$\frac{\partial S}{\partial t} + H\left(q, \frac{\partial S}{\partial q}, t\right) = 0$$

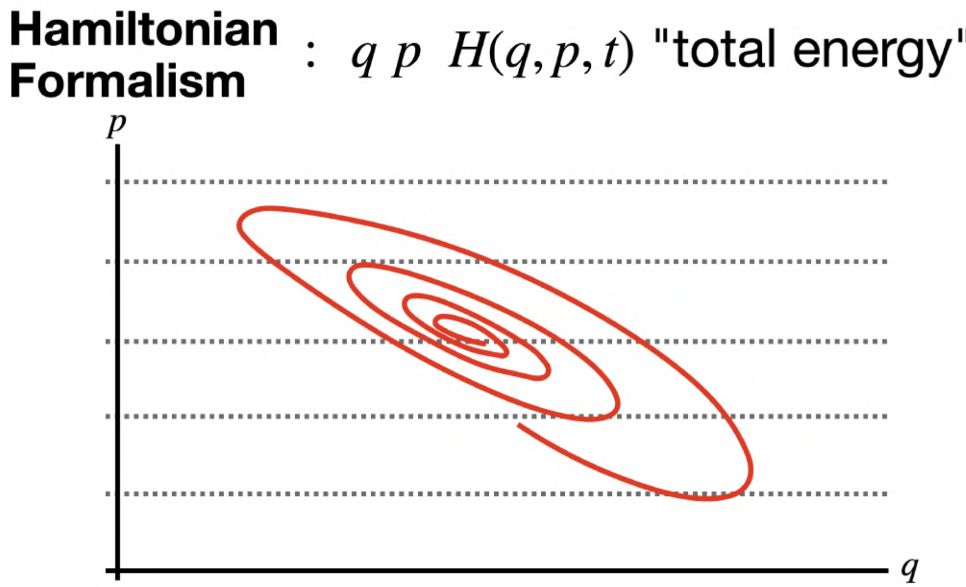
Hamiltonian function

The main difference between the Lagrangian and Hamiltonian formalisms is what they treat as fundamental variables.

The Lagrangian formalism works with position $q(t)$ and velocities $\dot{q}(t)$, which is useful for analyzing systems with *constraints* and *symmetries*. This is very useful when constructing *phase portraits* in dynamical systems, where the vertical axis is velocity and the horizontal one is position.



The Hamiltonian formalism, on the other hand, reformulates the problem in terms of position q and momenta p. These two quantities evolve according to the *Hamiltonian function* $H(q, p, t)$, which is often interpreted as the total energy of the system (in some contexts), and it determines the system's evolution in phase space (so, the space of all position-momentum pairs).



The Hamilton-Jacobi equation goes even further: it doesn't track trajectories directly. It stores all system's full dynamics into a single scalar function called action $S(q\,,\,t)$. Its gradients give us the momenta:

$$p_i = \frac{\partial S}{\partial q_i}(q\,,\,t)$$

Think of action as the core function that generates all the motion. Like a sort of potential for all trajectories.

Physicists tend to prefer the Lagrangian formalism (but not always). This is because it's usually more intuitive and practical for solving real-world problems. The equations of motion are usually easier to derive from a Lagrangian, and it also fits very well with QFT when using the principle of least action.

Mathematicians, on the other hand, often prefer the Hamiltonian formalism because of its abstract and geometric richness. It's a more general framework that gives a lot of insights into modern geometry. And it's especially useful when studying celestial mechanics, chaos theory and canonical transformations.

These are clearly not all the differential equations in Mathematical Physics, and making a list of the top 25 most useful ones is a difficult challenge. With that said, we did our best to select the ones that better represent the core of Mathematical Physics.

Please, let us know if we missed any: dibeos.contact@gmail.com

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`If you found this document useful let us know. If you found typos and things to improve, let us know as well. Your feedback is very important to us. We're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com`