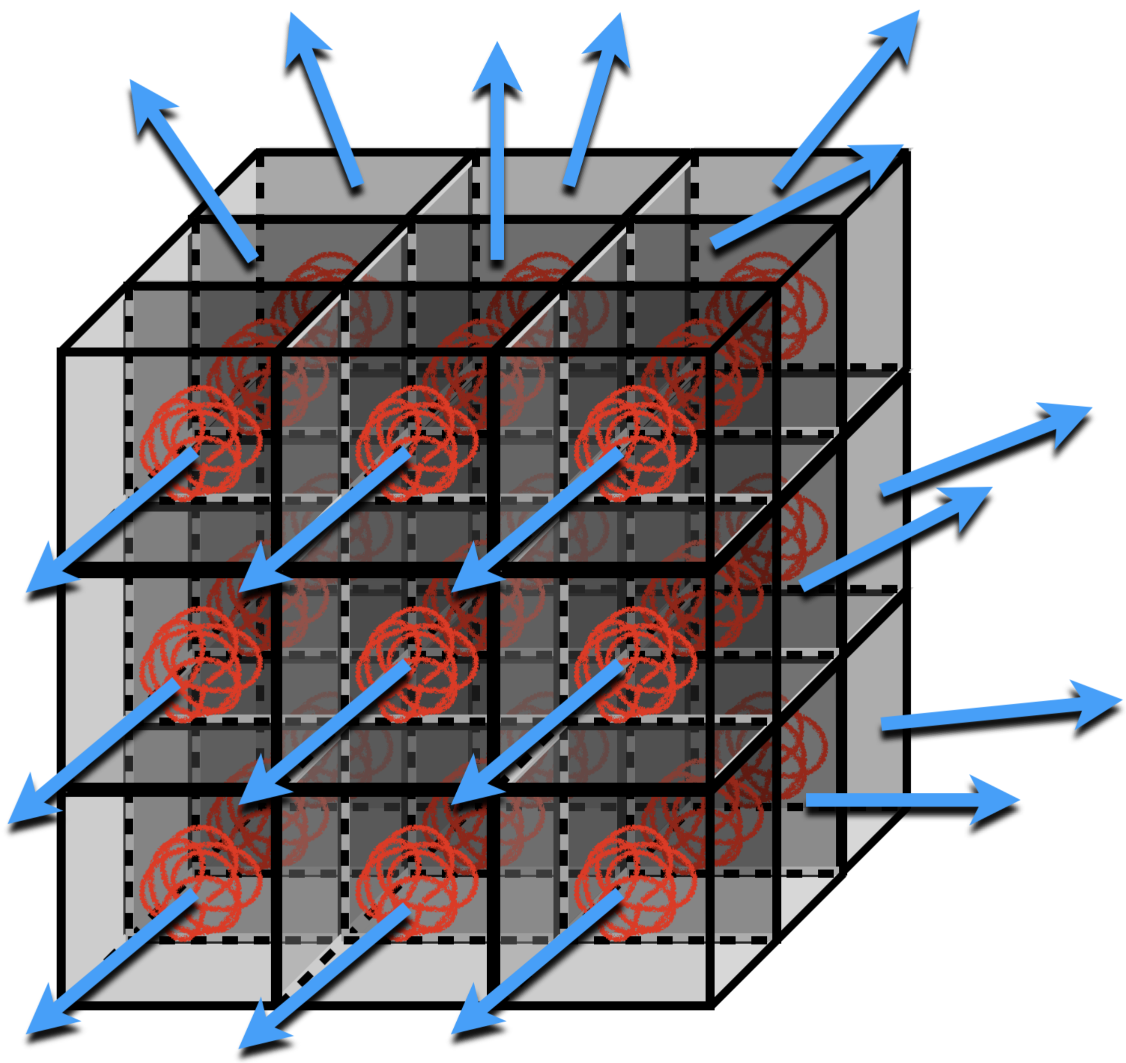
The Core of Tensor Calculus

by DiBeos



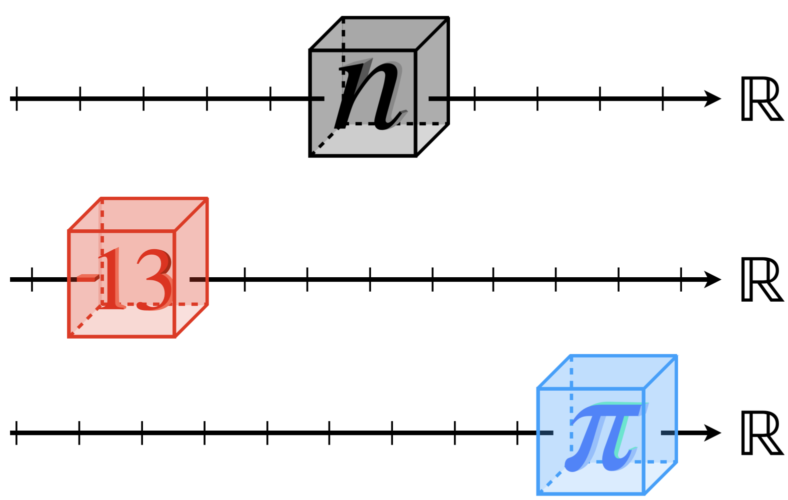
Introduction

Einstein cracked the code of spacetime not with numbers, but with tensors. But the truth is, tensors don't just describe physics. They are the geometry of the universe we live in.

In this PDF file you will learn what tensors are, and once you understand how they work, you'll never see space, time, geometry and motion the same way again.

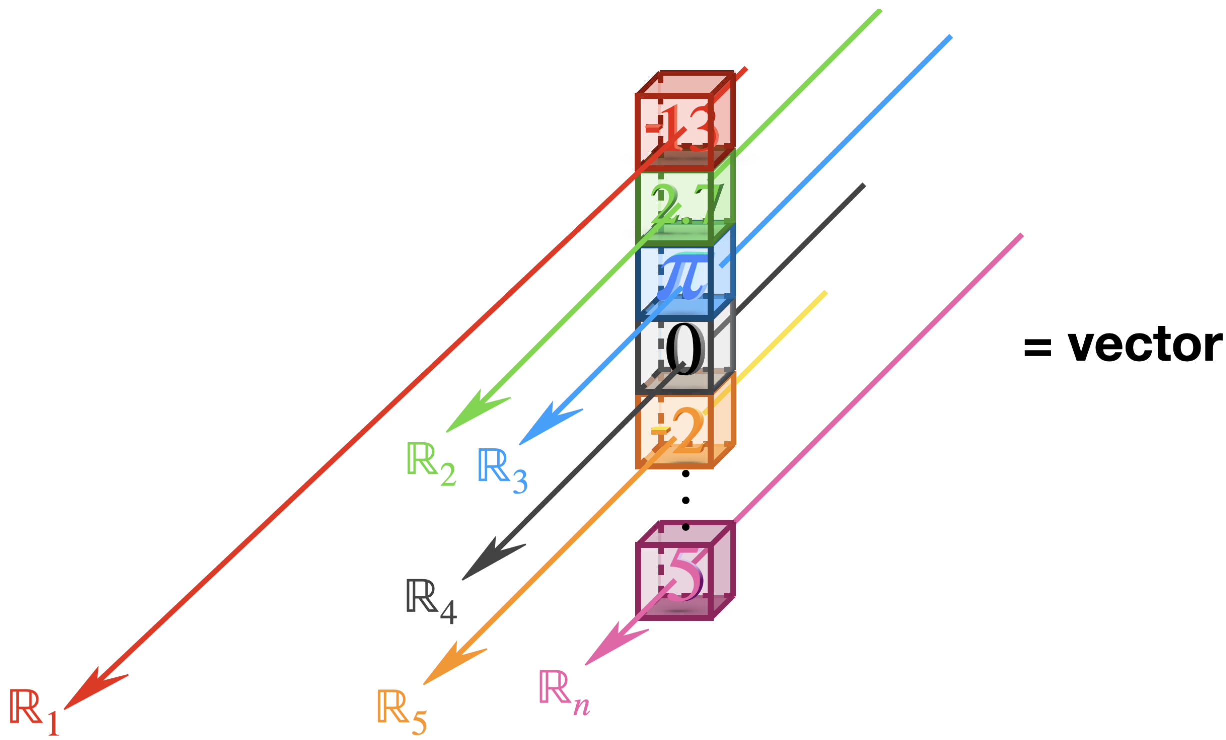
What is a tensor?

Imagine a scalar as just one number floating inside of a box. This value can change as you want, and the range of all possibilities is most often tracked by the real line.

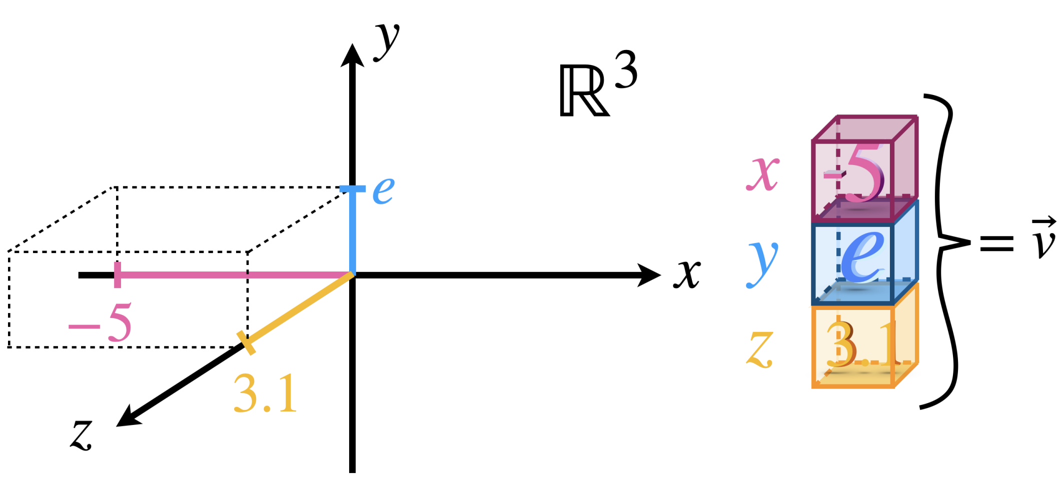


What about a vector?

One of its representations (but not the only one) is an array of scalars, each inside of their own box, lined up in a specific order. Each box can be tracked by its own version of the real line, for example.

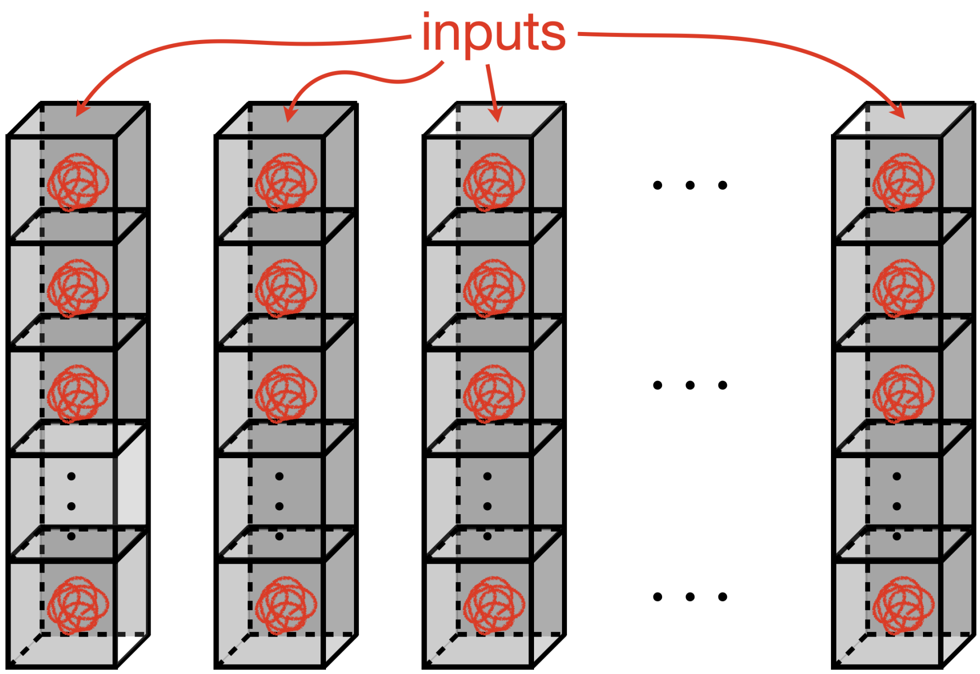


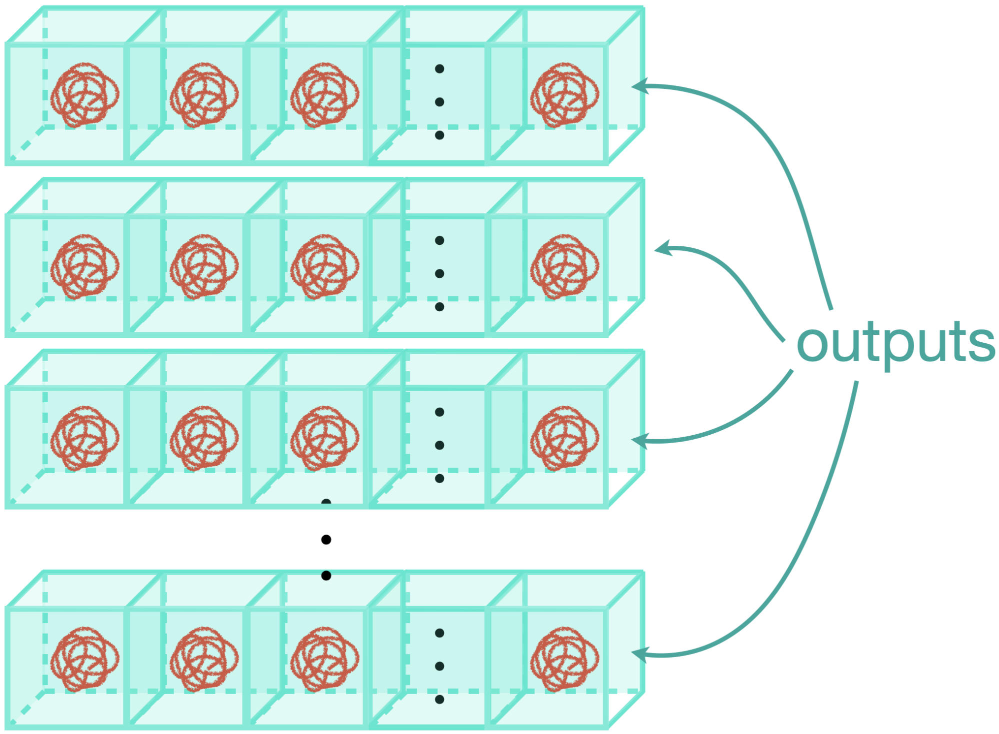
Each number tells you how far to go in a certain direction. In 3D, for example, you'd find 3 numbers, and their respective real lines are usually called (x, y, z). This is a vector: in this case, a 1D array of numbers. And the number of entries depends on the dimension of the space you're in.



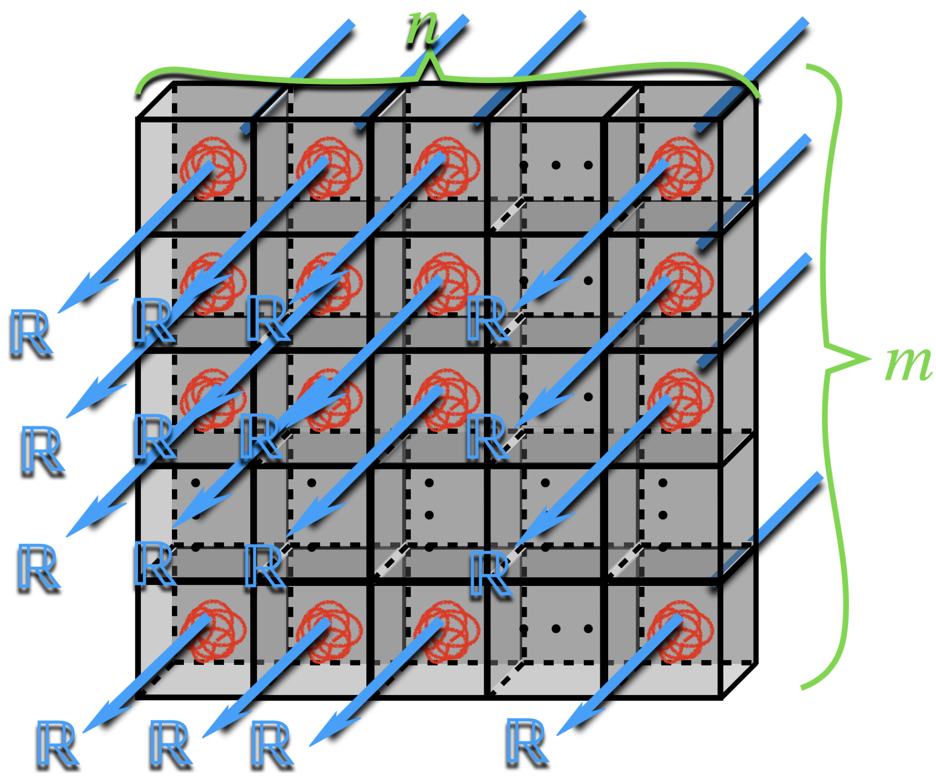
Using the same sort of analogy, what is a matrix?

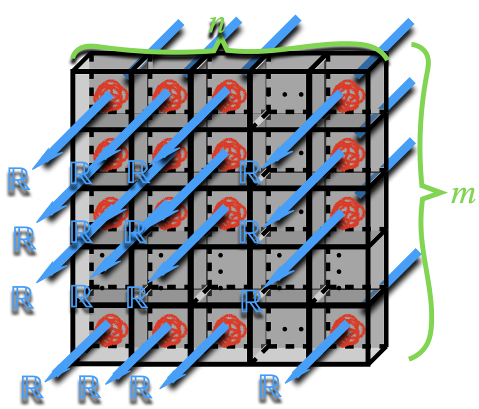
It can be represented as a table. Each column corresponds to an input direction, and each row to an output direction.

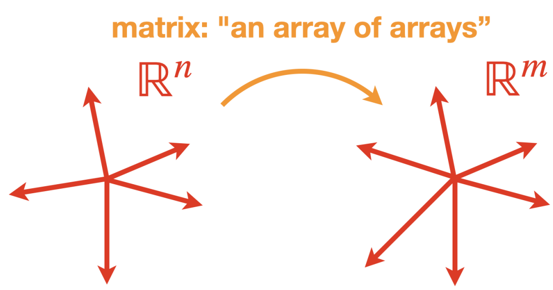




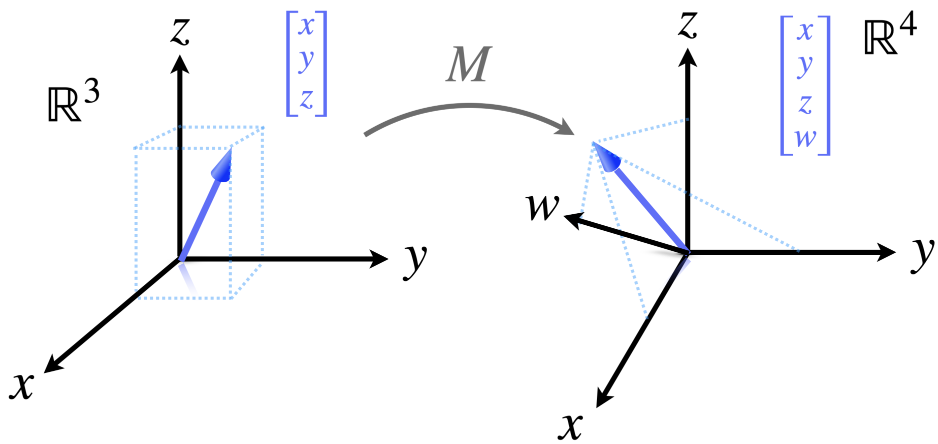
So, it maps vectors in an n-dimensional space $(\mathbb{R}^n)$ to vectors in an m-dimensional space $(\mathbb{R}^m)$. That's a matrix: "an array of arrays".



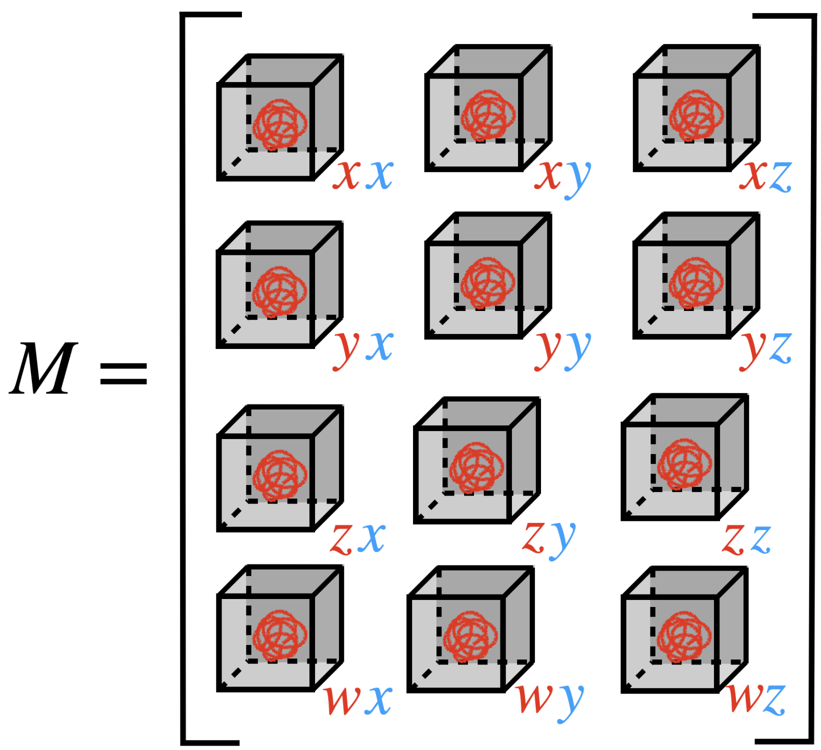




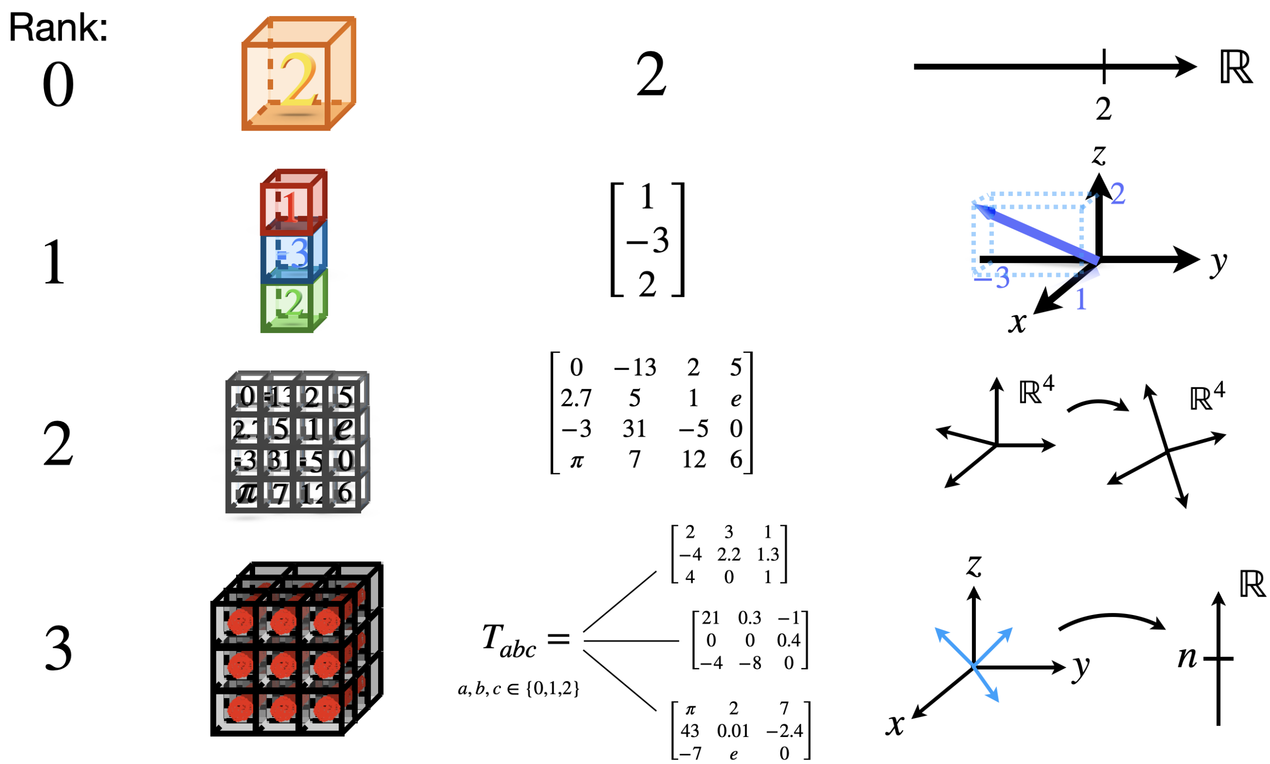
What happens if we continue to extend this idea to *arrays of arrays of arrays of arrays*... and so on? Notice that the number of layers of arrays has nothing to do with the dimension of the space we're working in. For example, think about an array of arrays, so a matrix M, that maps 3-vectors to 4-vectors.



This matrix maps a 3D space into a 4D space, however it is an array of arrays, i.e. just 2 *layers* (or *collections*) of arrays involved here. It has nothing to do with the dimensions of the underlying spaces.



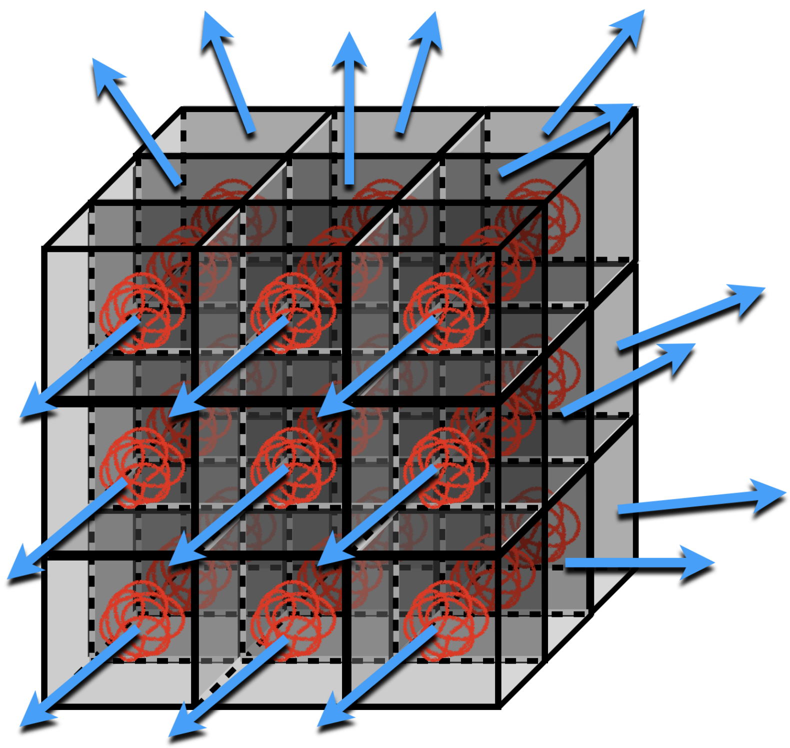
This "number of layers, or collections, of arrays" is called the **rank** (or **order**) of a tensor. And, by the way, all the mathematical objects that we've seen so far are tensors!



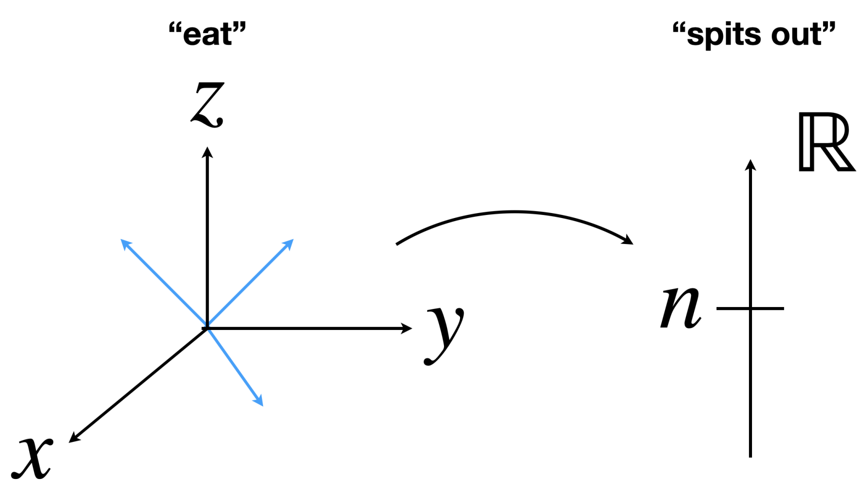
A vector is a *tensor of rank* 1, because there is only one array involved here. But a matrix is an array of arrays, so it's a *tensor of rank* 2. A scalar is a weird array because it contains only one entry, so it is not really an array. It has zero layers, or collections, of arrays. That's why we say that it is a *tensor of rank zero*.

There are also tensors with higher ranks, like a tensor with rank 3, which can be represented with a 3D cube of scalars. But, again, it's just a representation: the rank of a tensor has nothing to do with the dimension of the space it lives in.

It's too clumsy to draw here, but each of these entry boxes would have their own real lines sticking out of them as their range of possible values.



Continuing with the same logic, a tensor of rank 4 can be represented as a 4D cuboid. It's impossible to draw… but you get the idea.



Actually, not all tensors of rank 3 "eat" 3 distinct vectors (as shown here) and "spit out" a scalar. There are some tensors of rank 3 that "eat" vectors and "spit out" vectors too, or that "eat" and "spit out" covectors.

$$T^{abc} \qquad T^{a\ \ c}_{\ \ b} \qquad T_{a}^{\ \ bc}$$

$$T_{abc}$$

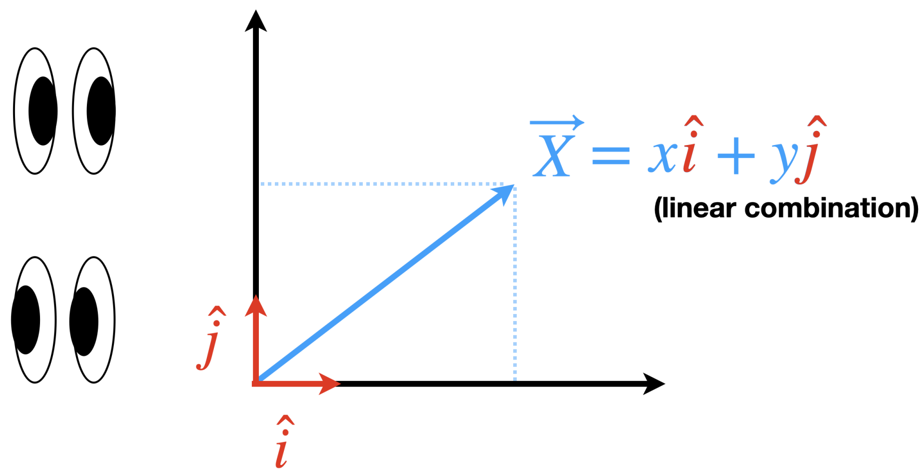
$$T^{ab}_{\ \ \ c} \qquad T_{a\ \ c}^{\ \ b} \qquad T^{a}_{\ \ bc}$$

$$T_{ab}^{\ \ \ c}$$

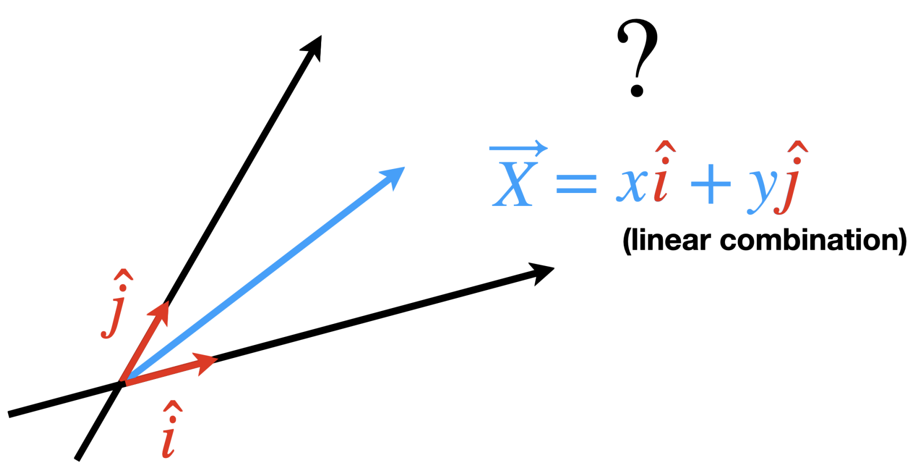
These distinctions come from the concepts of **covariant** and **contravariant** bases. But, before that… What is a covector, anyway?

What is a covector, anyway?

Well, think of a vector as an arrow on the real plane with its standard coordinate basis $\hat{\imath}$ and $\hat{\jmath}$, for example. The covectors are right there, staring at you, but you can't see them because this basis is orthogonal!

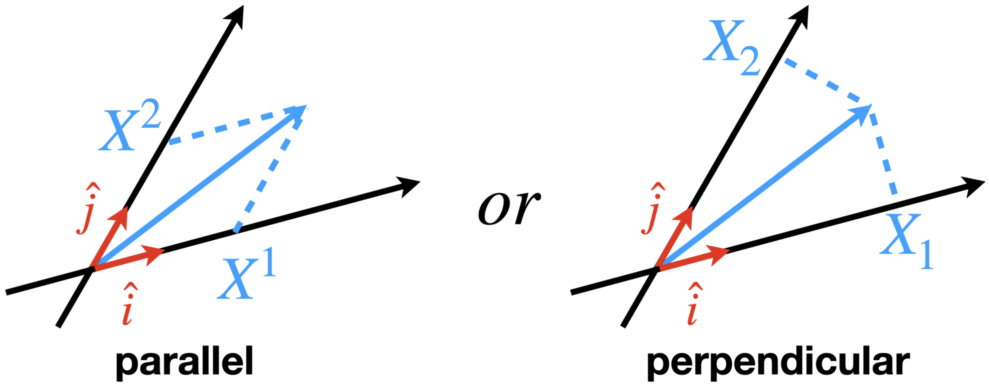


What happens, tough, if we tilt the basis axes this way (below)? Can we still describe the vector $\vec{X}$ as a *linear combination* of the elements of the basis?



The answer is YES, but we have two options now, so a *duality*.

Either we can decompose the vector $\vec{X}$ using dashed lines that are *parallel* to the basis axes, or using dashed lines that are *perpendicular* to the basis axes instead.



$$\vec{X} = X^1 \hat{i} + X^2 \hat{j} \qquad\qquad \vec{X} = X_1 \hat{i} + X_2 \hat{j}$$

Both representations are valid, but as you can see its components are different. In the first representation we chose to depict the components (which are just real numbers, by the way, not vectors) with SUPERscripts. In the second type of decomposition, we used SUBscripts instead.

SUPERscripts

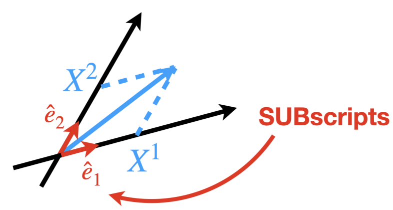
$$\vec{X} = \underset{\mathbb{R}}{\underset{\Cap}{X^1}} \hat{i} + \underset{\mathbb{R}}{\underset{\Cap}{X^2}} \hat{j}$$

$$\vec{X} = \underset{\mathbb{R}}{\underset{\Cap}{X_1}} \hat{i} + \underset{\mathbb{R}}{\underset{\Cap}{X_2}} \hat{j}$$

SUBscripts

Actually, the second representation (when written this way) is *wrong*, because the unit vectors $\hat{\imath}$ and $\hat{\jmath}$ are usually only used for the parallel projections. The question is: *Is there an alternative basis such that a linear combination of the elements of this new basis (with perpendicular projections) allows us to express the vector $\vec{X}$?* The answer is YES, and it's called: **the dual basis**.

Our original basis, defined by parallel projections, is usually denoted with $\hat{e}_1$ and $\hat{e}_2$ (SUBscripts) such that the little hats indicate that these are *unit vectors*. So, their lengths are 1. And it is called the *contravariant basis*. The vector components are X^1 and X^2 (SUPERscripts).

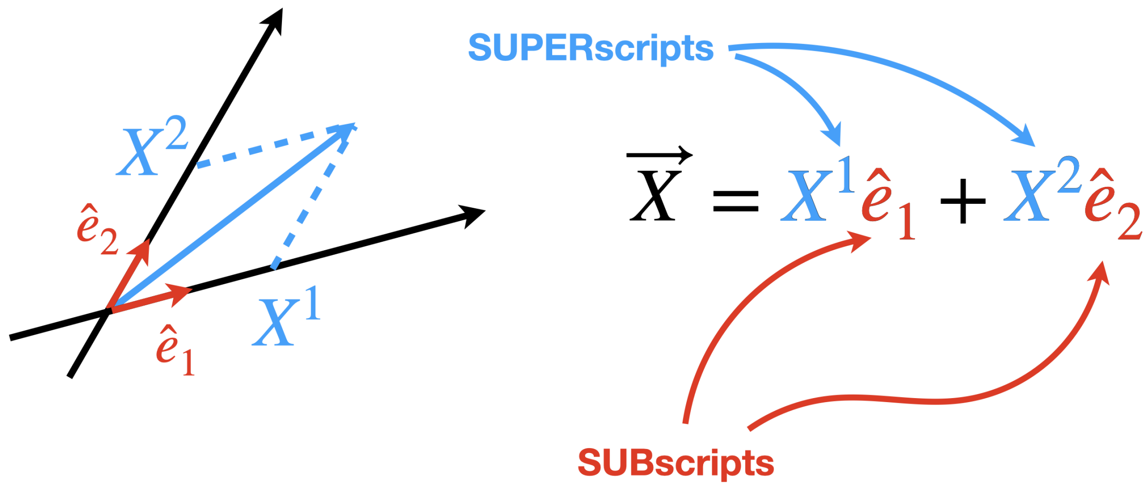


$$|\hat{e}_1| = |\hat{e}_2| = 1 \qquad \{\hat{e}_1, \hat{e}_2\} = \text{contravariant basis}$$

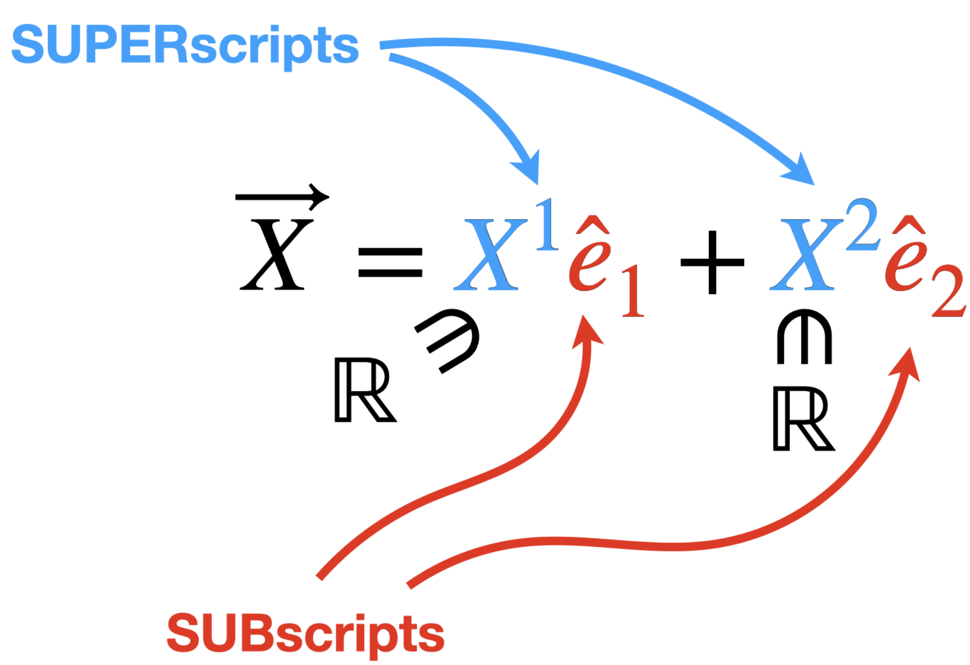
SUPERscripts

$$\vec{X} = X^1\hat{e}_1 + X^2\hat{e}_2$$

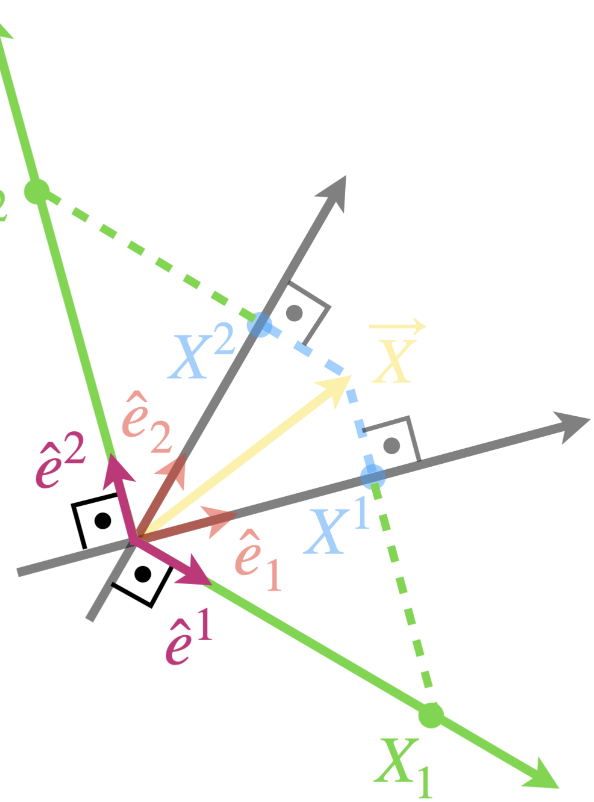
I know... it's kind of confusing since the basis vectors $\{\hat{e}_1,\ \hat{e}_2\}$ are written with indices DOWN, but the components of the vector are denoted with the indices UP. Don't worry about that yet. We will talk a little bit about "Index Notation and Einstein Summation" later on in this file. For now, do not pay much attention to whether the indices are up or down, just focus on distinguishing in your mind when we are dealing with a contravariant basis, opposed to the other type of basis (called covariant).

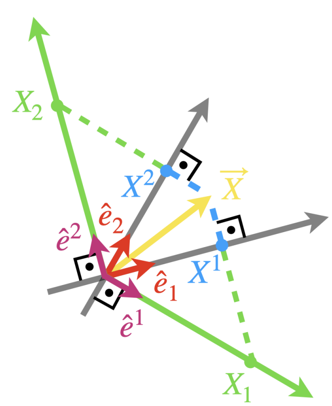


Also, it's important to remember that these components X^1 and X^2 are nothing but real numbers.



Now, its dual basis, defined by perpendicular projections on the original basis axes, can also represent the vector $\vec{X}$ by parallel projections on its own basis axes. These are denoted as $\hat{e}^1$ and $\hat{e}^2$ (SUPERscripts). This is called the *covariant basis*, and the vector components are real numbers denoted as X_1 and X_2 (SUBscripts).





SUPERscripts

$$\{\hat{e}^1, \hat{e}^2\} = \text{covariant basis} \qquad \vec{X} = X_1\hat{e}^1 + X_2\hat{e}^2$$

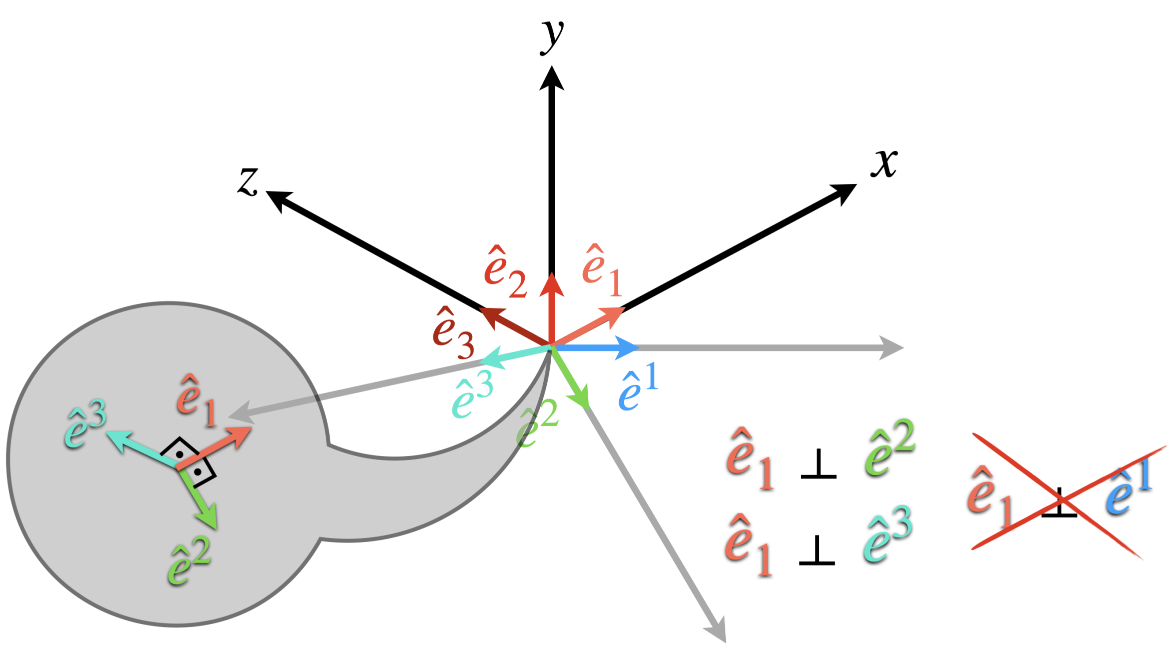
SUBscripts

Any two bases $\{\hat{e}_1, \hat{e}_2\}$ and $\{\hat{e}^1, \hat{e}^2\}$ are dual with respect to each other if they satisfy these conditions:

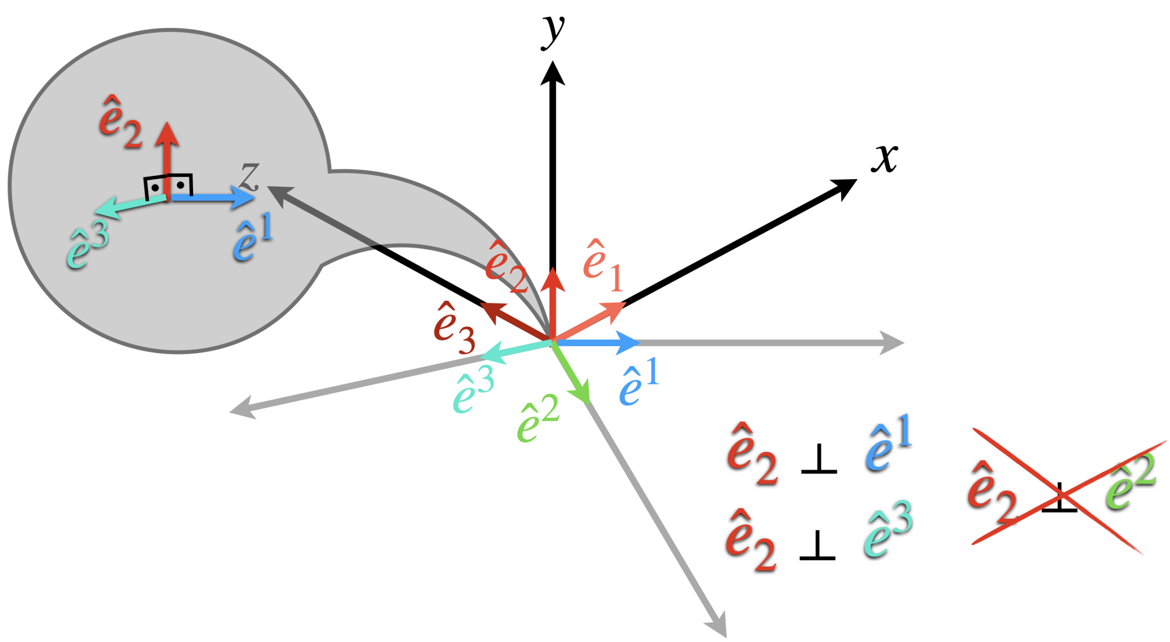
$$\{\hat{e}_1 \,,\, \hat{e}_2\} \xleftrightarrow{\quad \text{dual} \quad} \{\hat{e}^1 \,,\, \hat{e}^2\}$$

$$\hat{e}^j \cdot \hat{e}_i = \delta^j_i = \begin{cases} 1 & , \; i = j \\ 0 & , \; i \neq j \end{cases}$$

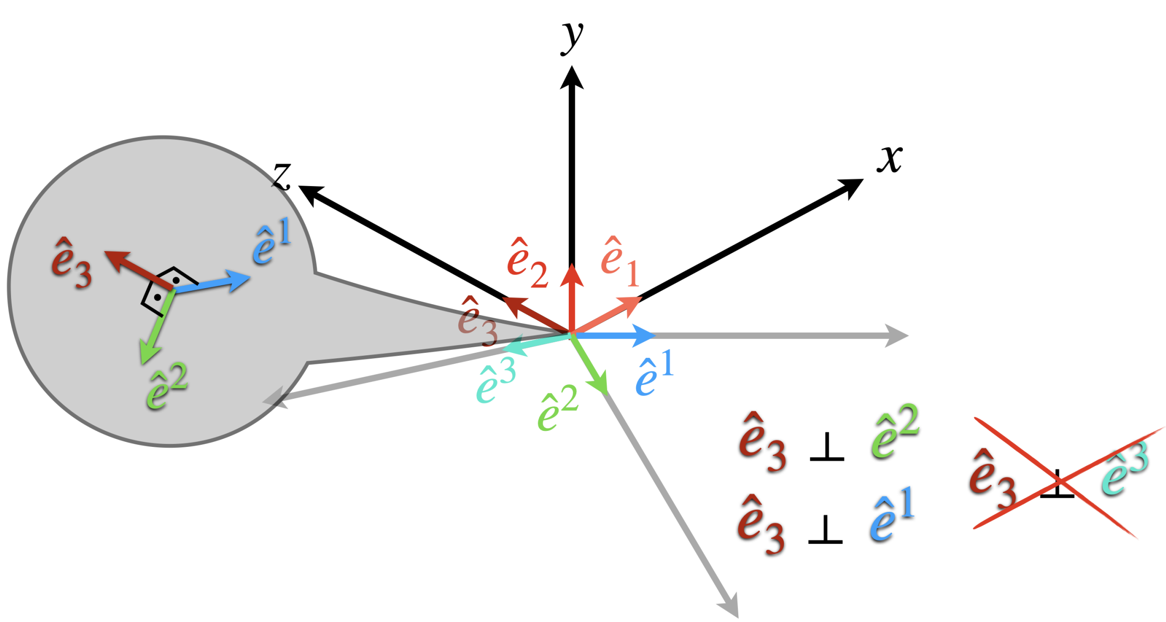
This means, for example in 3 dimensions, that $\hat{e}_1$ is orthogonal to $\hat{e}^2$ and $\hat{e}^3$, but <u>not</u> to its counterpart $\hat{e}^1$, i.e. its own dual.



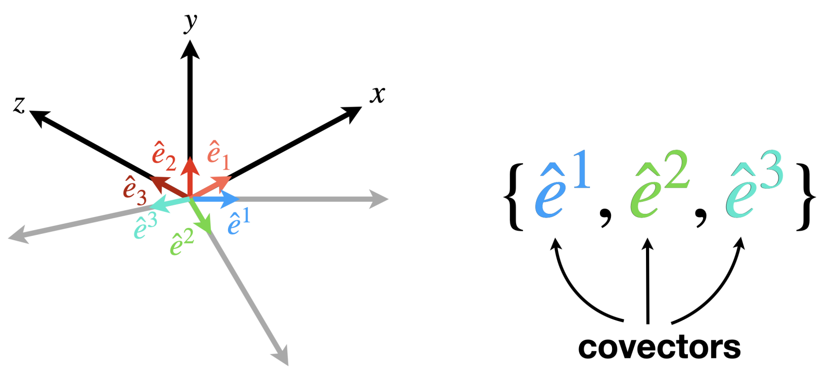
The same is valid for $\hat{e}_2$. It is orthogonal to $\hat{e}^1$ and $\hat{e}^3$, but <u>not</u> to its own dual $\hat{e}^2$.



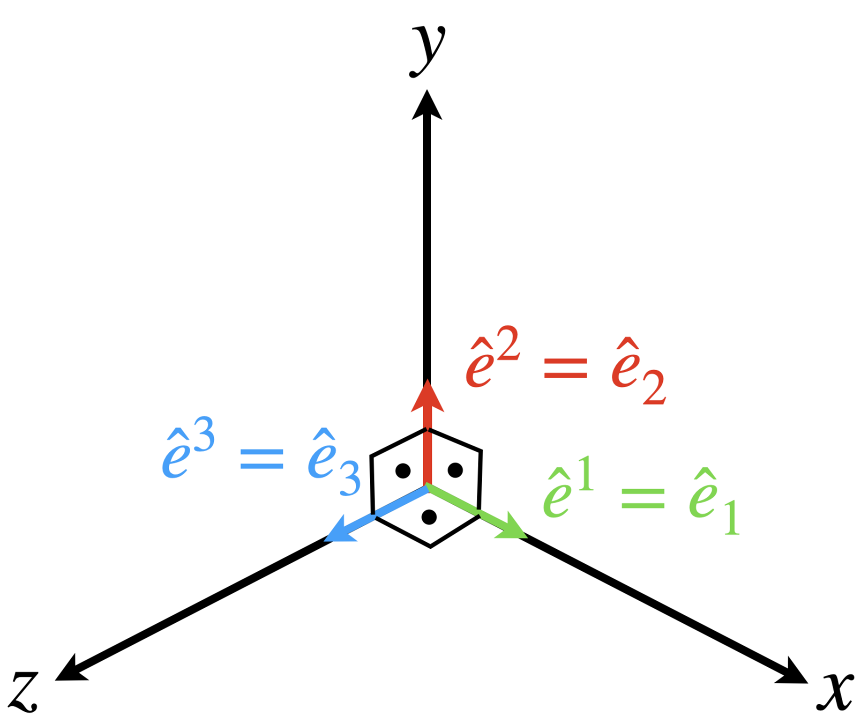
The same is true for $\hat{e}_3$.



These elements $\{\hat{e}^1, \hat{e}^2, \hat{e}^3\}$ in the dual basis are called **covectors**.



When the basis is orthogonal, the dual basis coincides with the original basis:



$\{\hat{e}^j \ : \ j = 1, ..., n\}$ can also produce, or describe, other covectors as well. If $\{\hat{e}_i\}$ is a basis for the vector space V, then $\{\hat{e}^j\}$ is its dual basis for its dual space V^*.

$$\underbrace{\left\{\hat{e}^j : j = 1, \ldots, n\right\}}_{\text{(covariant basis)}} \implies \text{other covectors}$$

$$\left\{\hat{e}_i\right\} \text{ basis for vector space } V$$

$$\Downarrow$$

$$\left\{\hat{e}^j\right\} \text{ basis for dual vector space } V^*$$

Any covector $\omega \in V^*$ can be written as:

$$\forall \underset{\text{covector}}{\omega} \in \underset{\text{dual space}}{V^*} : \qquad \begin{gathered} \omega = \omega_j \, \hat{e}^j \\ \text{or} \\ \omega = \omega_1 \, \hat{e}^1 + \omega_2 \, \hat{e}^2 + \ldots \end{gathered}$$

Using a similar logic we could talk about *co-matrices*, or *covariant matrices*, in a non-orthogonal coordinate system, and so on for other tensors of higher ranks. We won't discuss them in detail here, but I'll tell you that:

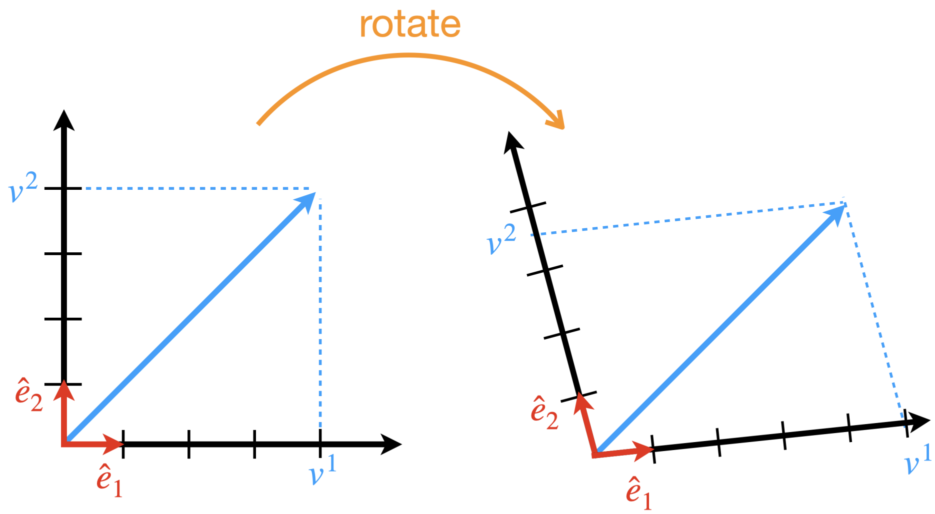
The **core of Tensor Calculus** lies exactly in understanding how geometric and physical quantities transform when the basis is non-orthogonal, which is something that happens very often especially when studying abstract curved spaces. Tensors provide the language to translate from contravariant to covariant basis, and vice-versa. So basically, from the original to the dual basis, back and forth. This is done by raising or lowering indices.

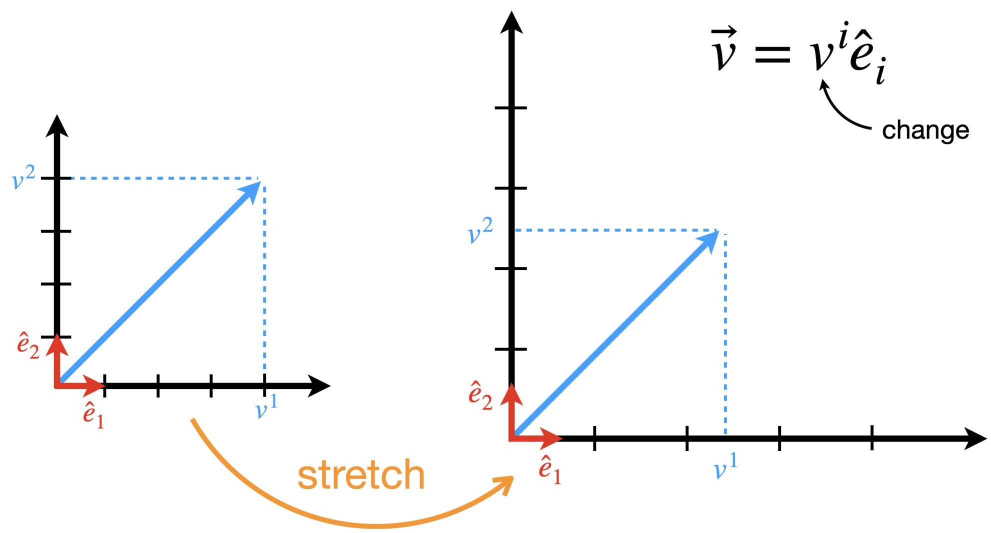
A contravariant vector is written with an upper index:

contravariant vector: v^i

$$\vec{v} = v^i \hat{e}_i$$

It transforms oppositely to the basis. This means that if you imagine a vector $\vec{v}$ as a physical arrow in space, and change the coordinate basis (say you rotate the basis axes or stretch them), then the components v^i in the vector $\vec{v} = v^i \hat{e}_i$ will change, but the physical arrow stays the same, invariant in space.

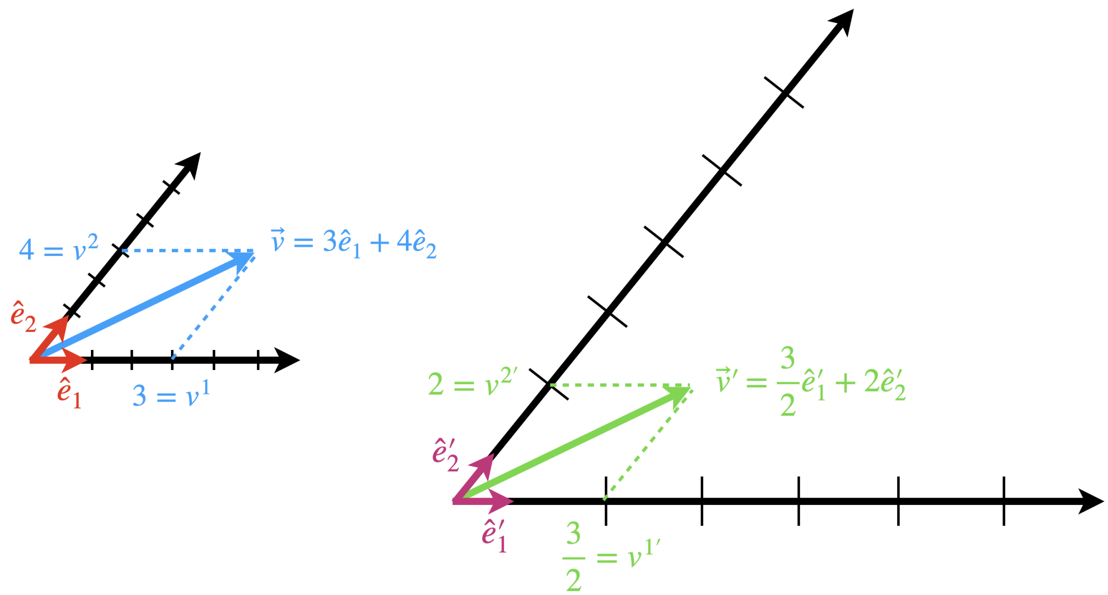




As you can see, contravariant vectors (like this one) are great to describe physical quantities, such as velocities, displacements, forces, and so on, because these things are not supposed to change in general just because you decided to use different "measuring sticks" (or even the same ones as before, but with different angles).

That's why you'll very often hear physicists saying things like: "contravariant vectors are more natural–they correspond to how geometric quantities exist independently of coordinates".

So, for example, say that the basis axes are scaled by a factor of 2, then the components v^1 and v^2 will be scaled inversely, i.e. by a factor of $\frac{1}{2}$.



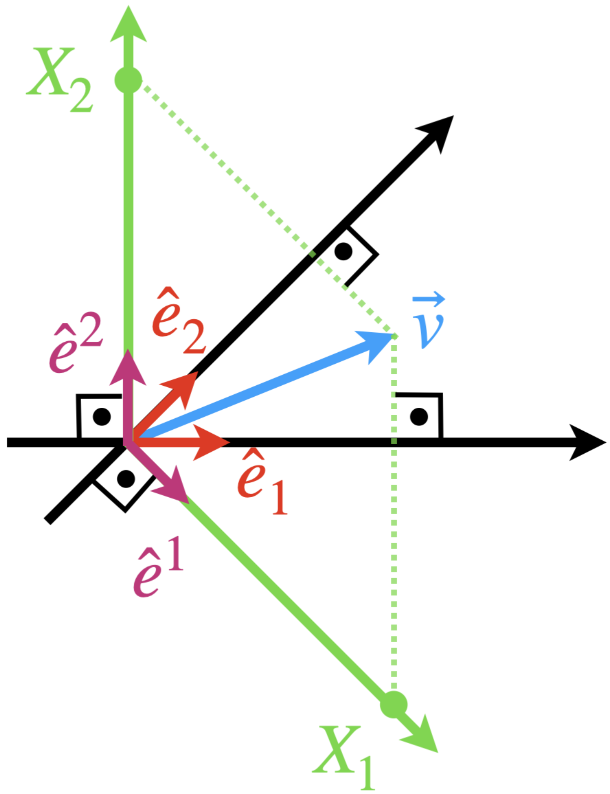
So, they transform opposite to the basis. This keeps the contravariant vector invariant:

$$\vec{v} = v^i \hat{e}_i$$

$$\vec{v} = v^{i'} \hat{e}'_i = \left(\frac{v^i}{2}\right) \left(2\hat{e}_i\right) = v^i \hat{e}_i$$

Great! But what about the basis covectors, i.e. the covariant basis vectors?

Imagine you have a vector space with non-orthogonal basis $\{\hat{e}_i\}$.



Its dual space has a basis $\{\hat{e}^j\}$. Since these bases are dual with respect to one another, by definition:

basis: $\{\hat{e}_i\}$

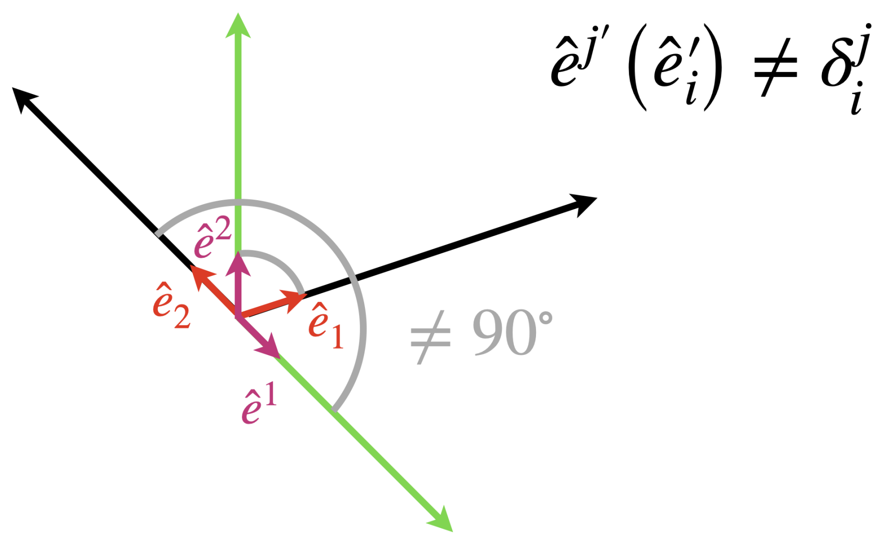
dual basis: $\{\hat{e}^j\}$

$$\hat{e}^j\left(\hat{e}_i\right) = \delta^j_i \quad \overset{\text{i.e.}}{\Longrightarrow}$$

$\hat{e}_1 \perp \hat{e}^2$	$\hat{e}_1 \not\perp \hat{e}^1$
$\hat{e}_2 \perp \hat{e}^1$	$\hat{e}_2 \not\perp \hat{e}^2$

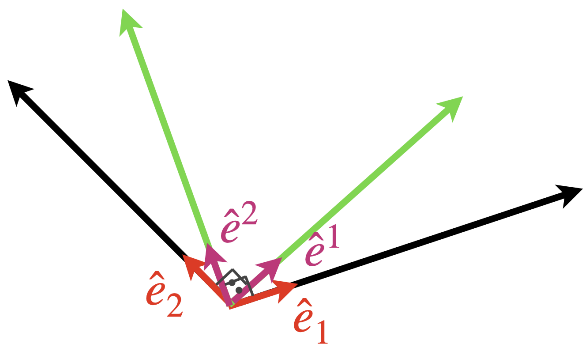
Now, suppose that we change the contravariant basis via a rotation.

The covectors in the dual basis $\{\hat{e}^j\}$ can't just be unchanged, otherwise the condition for duality would not be satisfied.



The new covariant basis still has to be orthogonal to the new contravariant basis (except for their individual counterparts, of course). And that's why we say that "the dual (or covariant) basis $\{\hat{e}^j\}$ transforms along/together with the original (or contravariant) basis $\{\hat{e}_i\}$, in order to maintain consistency."

$$\hat{e}^{j'}\left(\hat{e}'_i\right) = \delta^j_i$$



That's the goal of Tensor Calculus: to express invariant truths, even if the coordinate basis, and coordinate systems, change.

Therefore, T^i_{jk} (written this way) is a rank-3 tensor with 1 contravariant index, and 2 covariant indices. The only problem is that now we have a tensor that is not purely contravariant or purely covariant, but rather a mix of both.

2 covariant

$$T^{i}_{jk} = \text{rank} - 3 \ \text{tensor}$$

1 covariant

As a consequence its explicit version (in terms of basis vectors and covectors) looks like this:

$$T = T^i{}_{jk}\, \underbrace{\hat{e}_i}_{\text{vector}} \otimes \overbrace{\hat{e}^j \otimes \hat{e}^k}^{\text{covectors}}$$

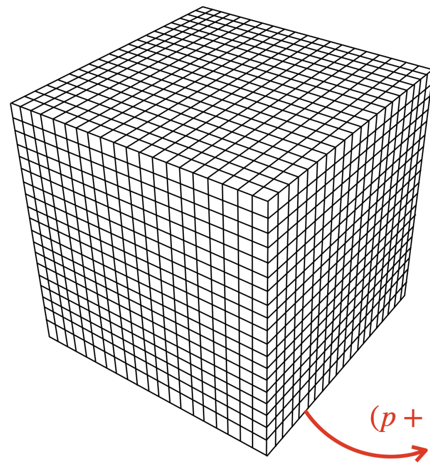
Here, this symbol $\otimes$ means **tensor product**, and it allows us to combine covariant with contravariant basis.

At this point, maybe some of you are asking yourselves: *What about a super-duper general tensor? What does it look like?*

$$\overset{\text{rank} - (p,q),\ p,q \in \mathbb{N}_0}{T} = T^{i_1\, i_2 \cdots i_p}_{j_1\, j_2 \cdots j_q}\ \hat{e}_{i_1} \otimes \ldots \otimes \hat{e}_{i_p} \otimes \hat{e}^{j_1} \otimes \ldots \otimes \hat{e}^{j_q}$$

This (right above) is a rank-(p,q) tensor, where p and q are natural numbers (including zero).
$T^{i_1 \ldots i_p}_{j_1 \ldots j_q}$ $\rightarrow$ remember, you can think of the components of this tensor organized in a sort of $(p+q)$-dimensional hypercube.

$$T^{i_1\, i_2\, \dots\, i_p}_{j_1\, j_2\, \dots\, j_q} =$$



Notice, for example, that in the case of a rank-$(1,1)$ tensor we write the first i as a SUPERscript, for its components, and for the corresponding basis vectors as a SUBscript.

SUPERscripts

rank – (1,1)

$$\boxed{T = T^{i}{}_{j}\, \hat{e}_i \otimes \hat{e}^j}$$

SUBscripts

Einstein Summation Convention

The same up and down convention is used for j. This is not a coincidence. It is the famous **Einstein Summation Convention**.

$$T = T^{i}{}_{j}\, \hat{e}_{i} \otimes \hat{e}^{j} = \sum_{i=1}^{n} T^{i}{}_{j}\, \hat{e}_{i} \otimes \hat{e}^{j} =$$

Every time you see the same index, appearing twice, one of them up and the other down, it means that you should sum them up, and the tensor is said to be contracted in this index.

$$T = T^{i}{}_{j}\, \hat{e}_{i} \otimes \hat{e}^{j} = \sum_{i=1}^{n} T^{i}{}_{j}\, \hat{e}_{i} \otimes \hat{e}^{j} = \sum_{j=1}^{m} \sum_{i=1}^{n} T^{i}{}_{j}\, \hat{e}_{i} \otimes \hat{e}^{j} =$$

$$= \sum_{j=1}^{m} \left(T^{1}{}_{j}\, \hat{e}_{1} \otimes \hat{e}^{j} + T^{2}{}_{j}\, \hat{e}_{2} \otimes \hat{e}^{j} + \ldots + T^{n}{}_{j}\, \hat{e}_{n} \otimes \hat{e}^{j} \right) =$$

$$= \left(T^{1}{}_{1}\, \hat{e}_{1} \otimes \hat{e}^{1} + \ldots + T^{1}{}_{m}\, \hat{e}_{1} \otimes \hat{e}^{m} \right) + \left(\ldots \right) +$$

$$+ \left(T^{n}{}_{1}\, \hat{e}_{n} \otimes \hat{e}^{1} + \ldots + T^{n}{}_{m}\, \hat{e}_{n} \otimes \hat{e}^{m} \right)$$

$$\boxed{\text{CONTRACTED}}$$

Yeah, as you can see, writing this tensor explicitly results in a huge expression, and that's why some people say that "Einstein's greatest contribution to Pure Mathematics was his summation convention".

$$T^{i}{}_{j}\, \hat{e}_{i} \otimes \hat{e}^{j}$$

Einstein went even further by informally referring to tensors simply using their components $T^i{}_j$. This perspective became so intuitive and practical (especially in physics) that it eventually influenced even pure mathematicians working with tensors.

$$T^i{}_j$$

Unfortunately we won't have time to see all the important things about tensors. But this is a list of the main concepts that anyone who's serious about learning **Tensor Calculus** should know:

1) Scalars, vectors, matrices, and tensors

2) Index notation and Einstein summation

3) Coordinate transformations

4) Metric tensor

5) Raising and lowering indices

6) Covariant derivative

7) Christoffel symbols

8) Riemann curvature tensor

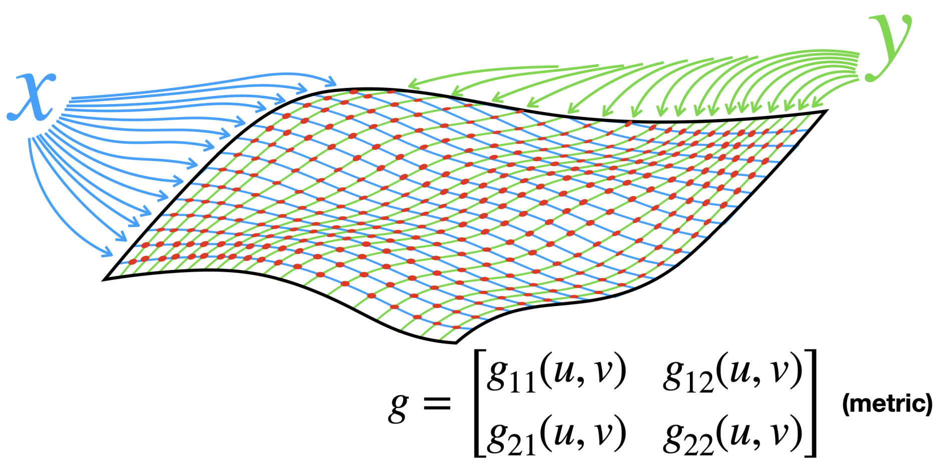
9) Ricci tensor and scalar curvature

Let us know if you guys would like videos and PDF files dedicated to some of these subjects: dibeos.contact@gmail.com

Ok, to conclude the video we'll quickly see some very useful examples of tensors that you've probably encountered in many different contexts. And if you've never seen them before, you might want to pay close attention because they show up often in pure and applied areas of math.

1 The Metric Tensor g_{ij}

Type: rank-$(0, 2)$

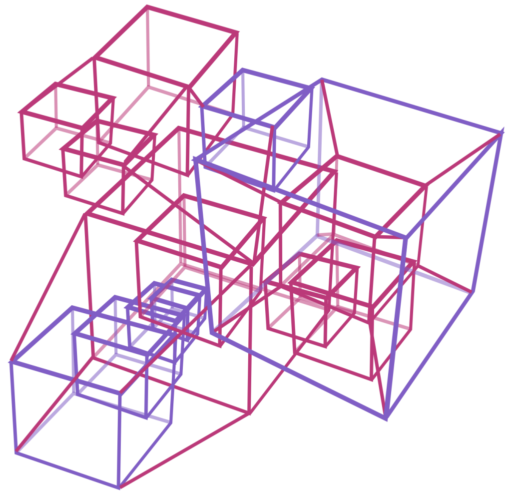


It measures *distances* and *angles* on a manifold, like Riemannian metric tensors, or even pseudo-Riemannian ones – just as in General Relativity.

2 The Riemann Curvature Tensor R^i_{jkl}

Type: rank-$(1, 3)$

$$R^{l}_{ijk} \quad (nD)$$



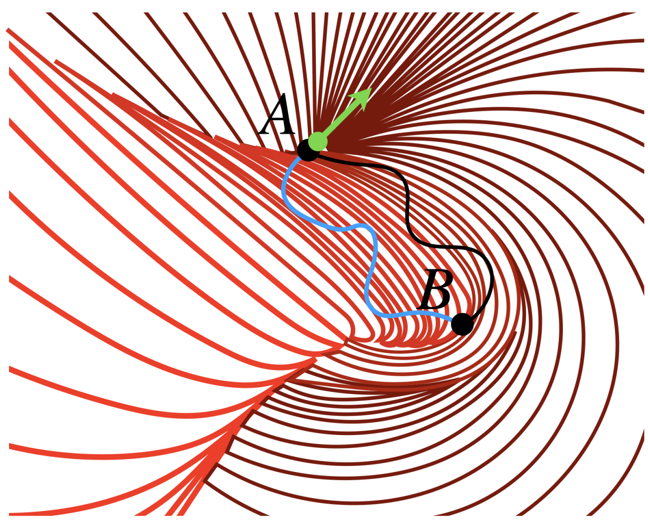
This tensor encodes intrinsic curvature of a manifold at each point.

3 The Torsion Tensor T^{i}_{jk}

Type: rank-$(1, 2)$

$$T^{i}{}_{jk} = \Gamma^{i}{}_{jk} - \Gamma^{i}{}_{kj}$$

Christoffel symbols

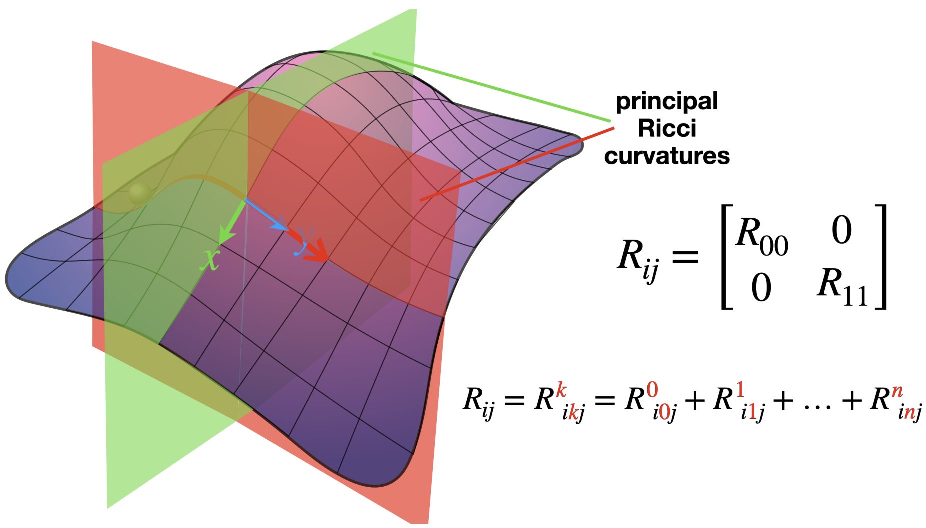


This tensor measures how much "twist" happens when you try to move vectors around in space. I.e., it measures if moving from A to

B results in a different direction compared to moving from B to A–in which case the torsion tensor is *not trivial* (i.e. not zero in all its components at the same time).

4 The Ricci Tensor R_{ij}

Type: rank-$(0,2)$



It's the contraction of the Riemann tensor on the first and third indices. As a consequence, it is purely covariant. And it encodes how volumes distort under curvature.

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