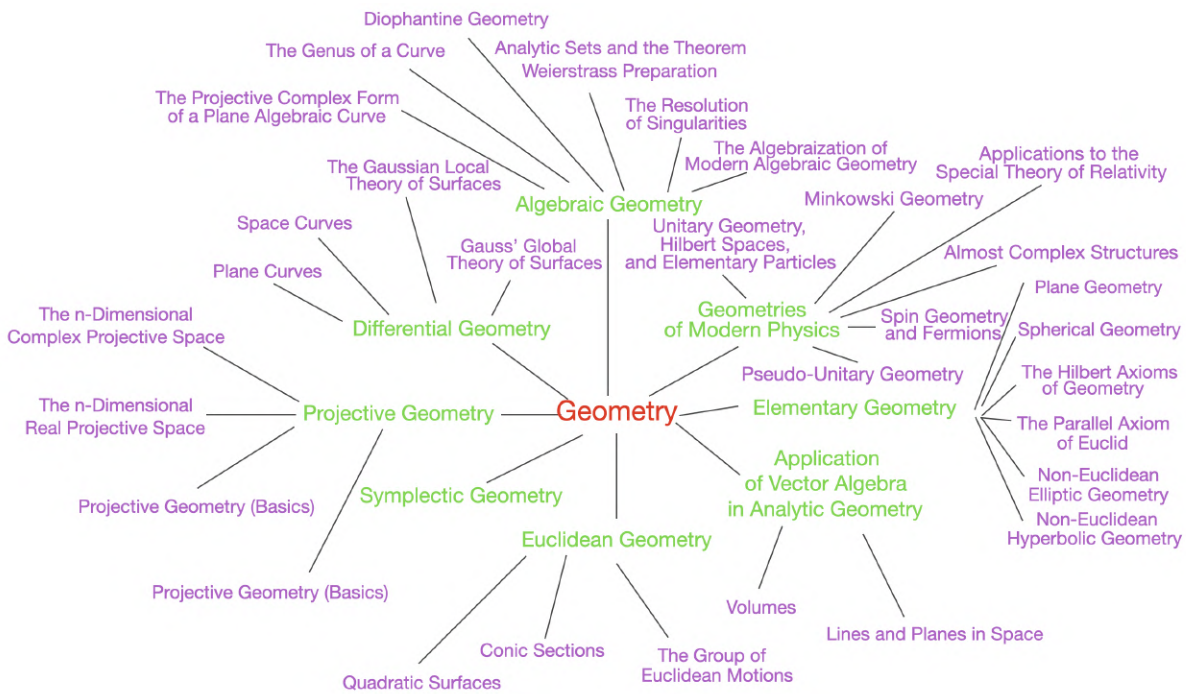
Every Concept in Geometry

by DiBeos

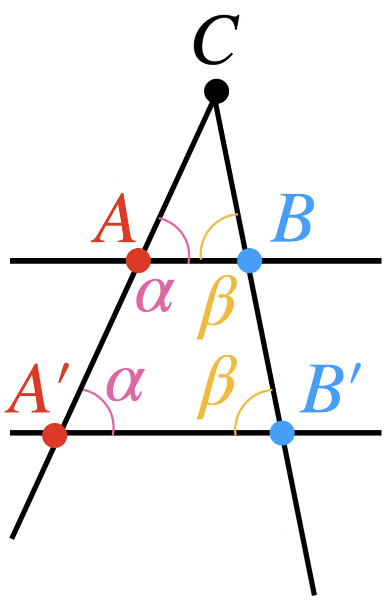


Geometry started with intuitive Euclidean concepts, but underwent a major transformation when we opened up the world of non-Euclidean geometries. Let's see what these concepts are and how they evolved.

We start with **Elementary Geometry**:

Plane Geometry

Plane geometry begins with *trigonometry*, which is the foundational field for understanding *angles* and *lengths* in triangles. It deals with concepts like *sine*, *cosine*, and *tangent* functions and their laws.



$$\frac{CA}{CA'} = \frac{CB}{CB'}$$

Basically, all kinds of concepts and theorems that describe relationships between angles and sides of triangles, their *areas*, and relationships with other *polygons* and *circles*.

Law on the sum of the angles:

$$\alpha + \beta + \gamma = \pi$$

Cosine law:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

Sine law:

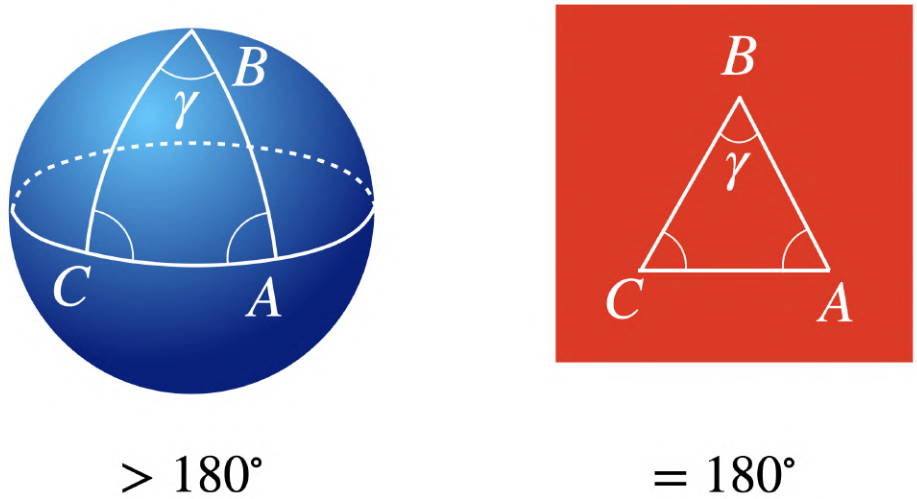
$$\frac{a}{b} = \frac{\sin\alpha}{\sin\beta}$$

Tangent law:

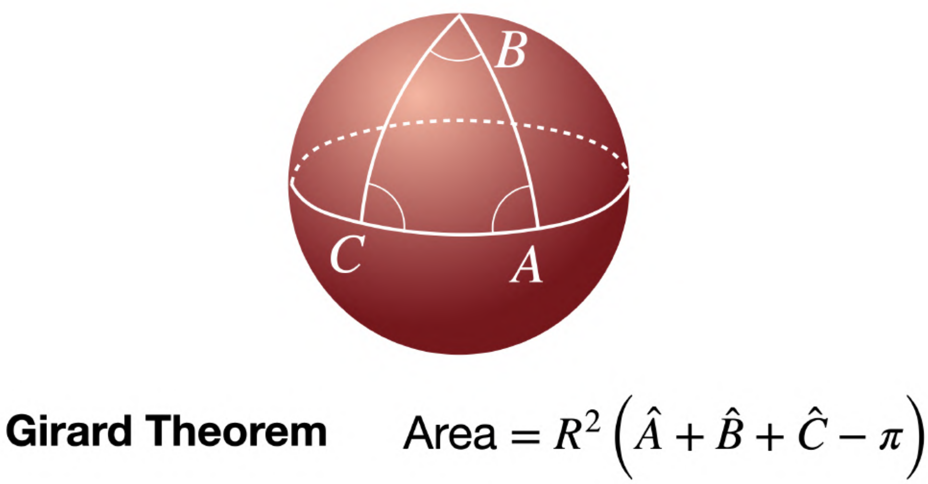
$$\frac{a-b}{a+b} = \frac{\tan\frac{\alpha-\beta}{2}}{\tan\frac{\alpha+\beta}{2}} = \frac{\tan\frac{\alpha-\beta}{2}}{\cot\frac{\gamma}{2}}$$

Spherical Geometry

The field studies figures on the surface of a *sphere*, and its basic elements are quite different from those of Euclidean geometry. The main idea is that the shortest path between two points on a sphere lies along a *great circle*. This would be analogous to a straight line, but on a curved surface, like *spherical triangles* for example, which are different from planar triangles because the sum of the interior angles of a spherical triangle is greater than π, or 180°.

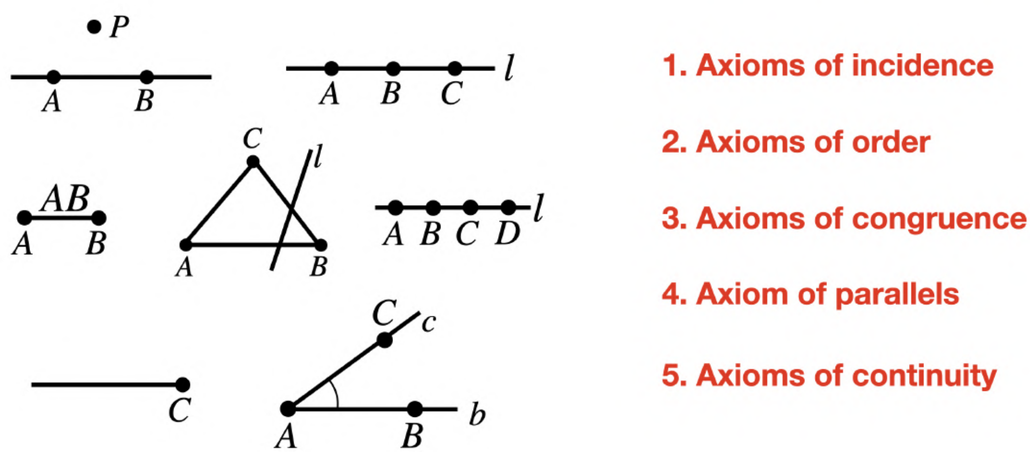


The excess (called the *spherical excess*) is directly related to the area of the triangle on the sphere, and this leads us to the *Girard theorem*: Area $= R^2\,(A + B + C - \pi)$, where A, B, C are the angles of the spherical triangle and R is the radius of the sphere.



The Hilbert Axioms of Geometry

David Hilbert was the one who rigorously reconstructed *Euclidean geometry* using a formal axiomatic system, so that the system of geometry was based on a foundation of *pure logic*. He established three notions that are to be the start: *point*, *line*, *incident*, *between*, and *congruent*.



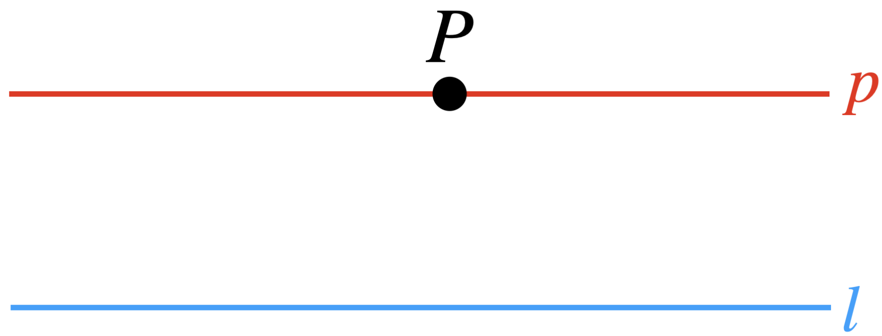
Hilbert's axioms are a set of 20 assumptions, but they can be grouped into five categories:

1. *Axioms of incidence*
2. *Axioms of order*
3. *Axioms of congruence*
4. *Axiom of parallels*
5. *Axioms of continuity*

The Parallel Axiom of Euclid

This is known as **Euclid's Fifth Postulate**, often called the *parallel postulate*. It (loosely speaking) states that given a line, and a point not on the line, there is exactly one line through the point that does not intersect the given line, also known as its *parallel*.

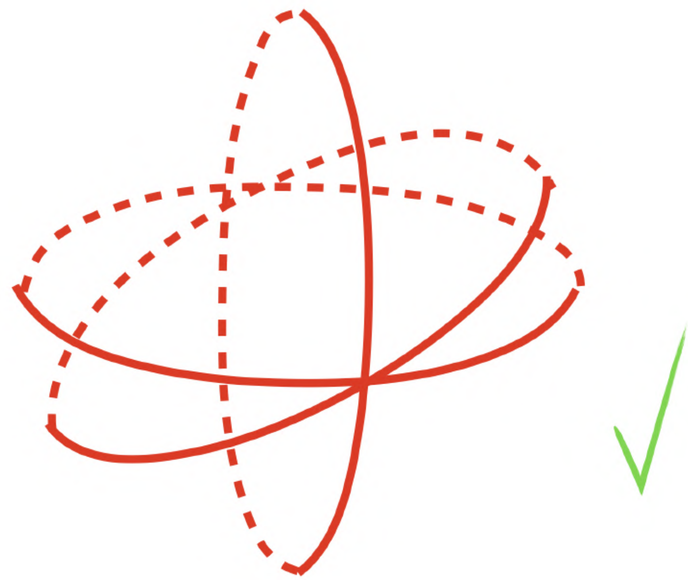
Euclid's Fifth Postulate



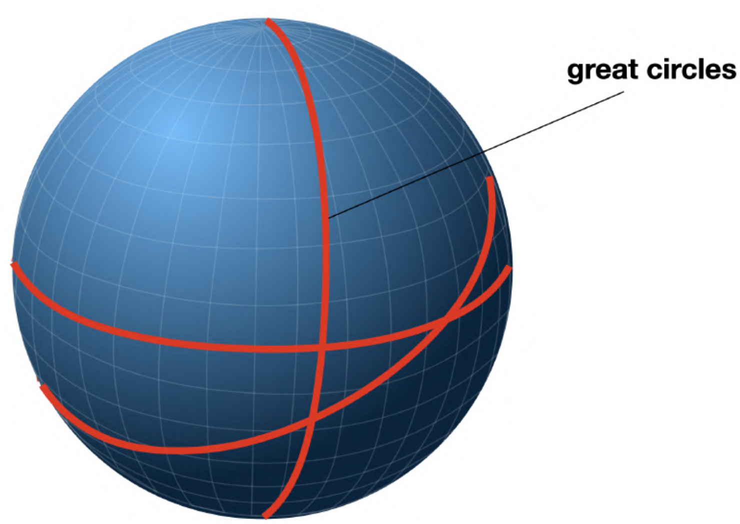
Oddly enough, even though there are other 4 postulates of Euclid's, this one took centuries of attempts to actually prove. And, rejecting it leads to *non-Euclidean geometries*.

Non-Euclidean Elliptic Geometry

Now, if we replace Euclid's parallel axiom, with the statement that, for example, *no parallel lines exist* (or in other words that every pair of lines intersects), we get a completely different, and yet entirely consistent type of geometry.



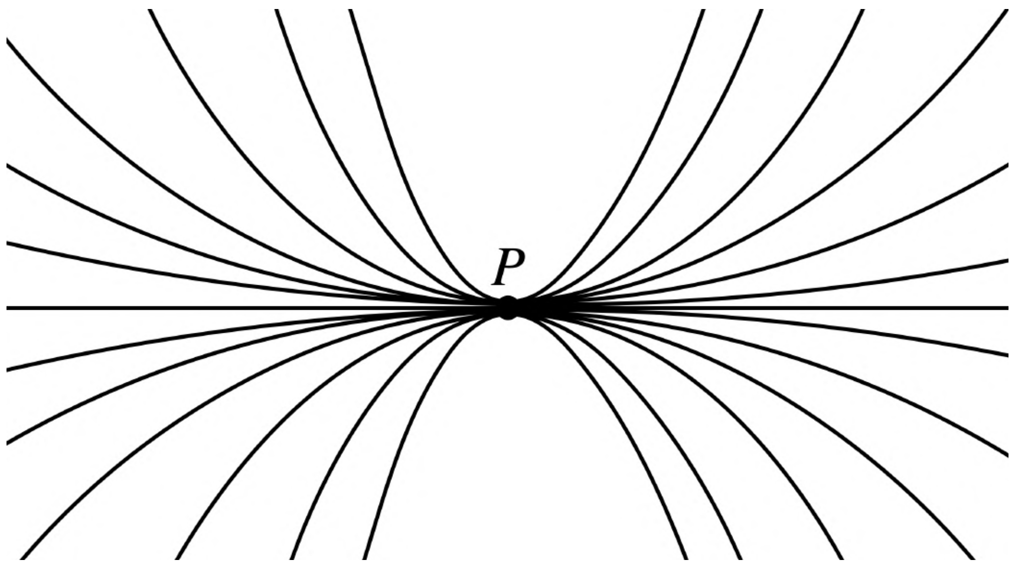
On the surface of a sphere and in this setting, lines are now great circles, the sum of angles is greater than π, there are *no parallels*, *no infinite lines*, and *no absolute distance to infinity*.



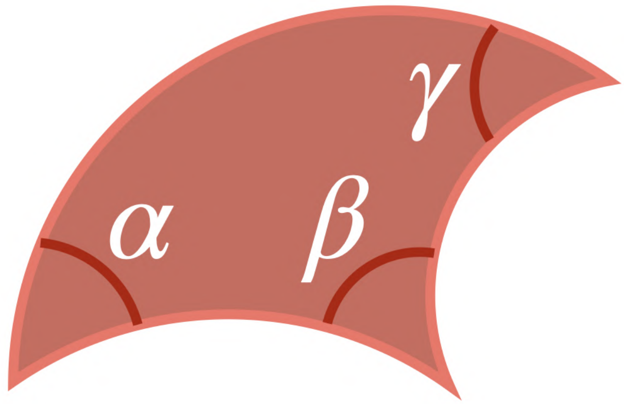
This is especially useful in studying *closed cosmological models*, for example.

Non-Euclidean Hyperbolic Geometry

Let us now assert that through a point not on a given line, there exist *infinitely many lines* that do not intersect the given line. So there can be *multiple parallels*.

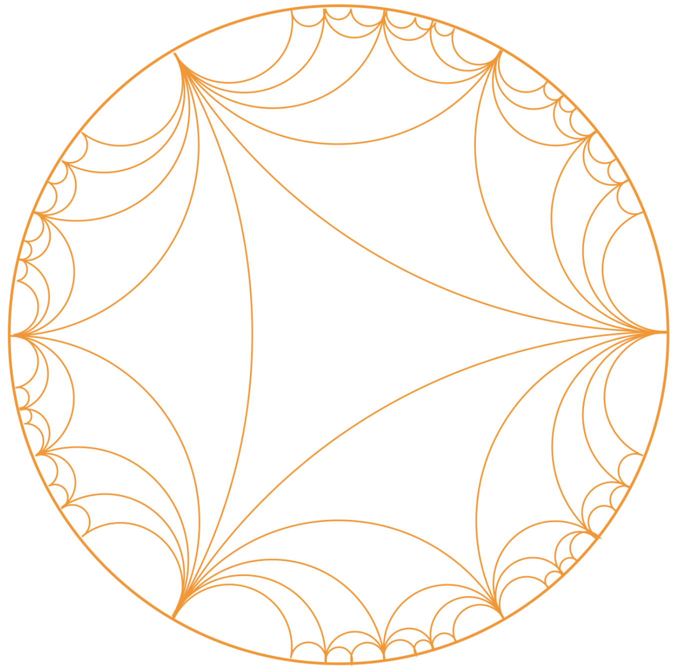


This means for example, that the sum of angles in triangles is now less than π.



$$\alpha + \beta + \gamma < \pi$$

Some models that represent **hyperbolic geometry** are the *Poincaré disk model* and the *hyperboloid model*, where “lines” appear as *arcs* or *curves*, but still obey the strict axiomatic rules.



Poincaré disk model

Now we move on to a different type of geometry, which is:

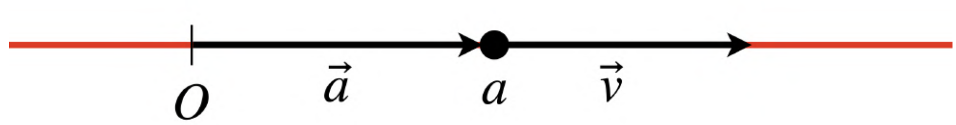
The Applications of Vector Algebra in Analytic Geometry

The **applications of Vector Algebra in Analytic Geometry** lets us solve geometric problems in both the *plane* and the 3-*dimensional space*. Points can be now represented as *vectors*, and relations like lines, planes, distances, and angles, are translated into *dot products* and *cross products*.

Lines and Planes in Space

Lines and planes can be translated into 3-dimensional spaces using vector notation.

$$\vec{x}(t) = \vec{a} + t\vec{v}$$

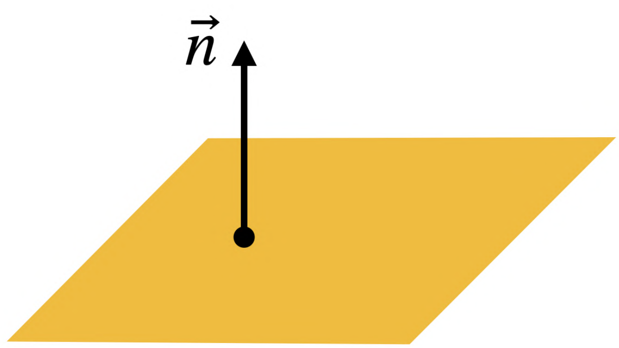


For example a line is described parametrically as

$$\vec{x}(t) = \vec{a} + t\vec{v}$$

where $\vec{a}$ is a point on the line and $\vec{v}$ is its direction vector.

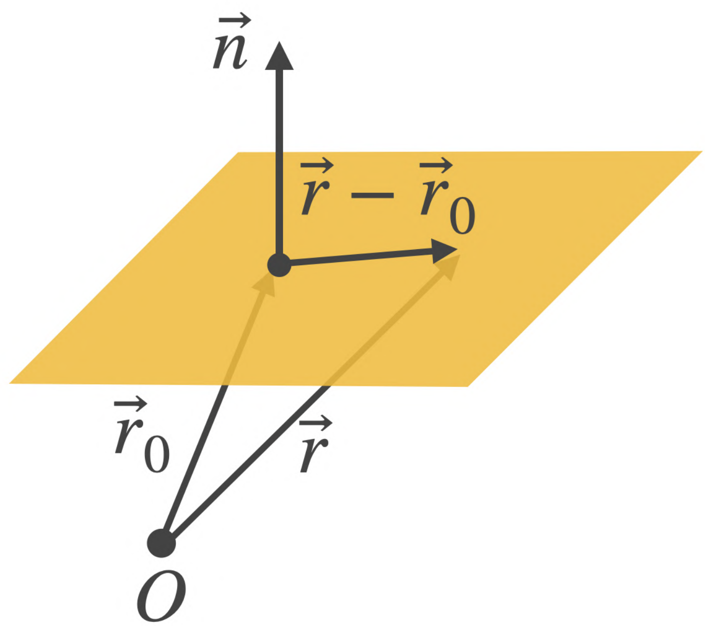
$$(\vec{x} - \vec{a}) \cdot \vec{n} = 0$$



A plane though, is defined by the scalar equation

$$(\vec{x} - \vec{a}) \cdot \vec{n} = 0$$

where $\vec{n}$ is a normal vector to the plane.



Vector algebra allows us to compute *intersections*, *distances*, and *angles* between lines and planes.

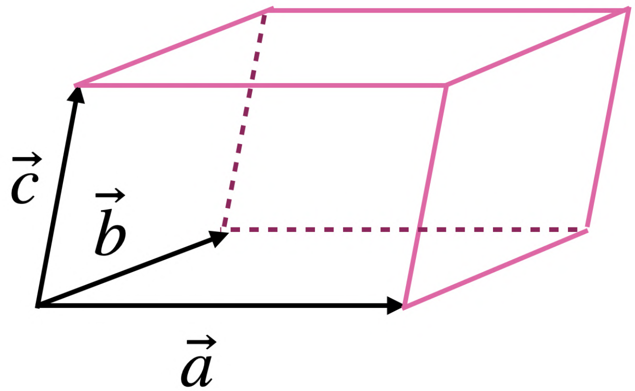
Volumes

When it comes to *geometric solids*, in order to compute their **volume** we have to use vector methods. One of the key tools is the scalar triple product, for example:

$$V = \left|\vec{a} \cdot \left(\vec{b} \times \vec{c}\right)\right|$$

which gives the *volume of the parallelepiped*, spanned by three vectors $\vec{a}, \vec{b}, \vec{c}$.

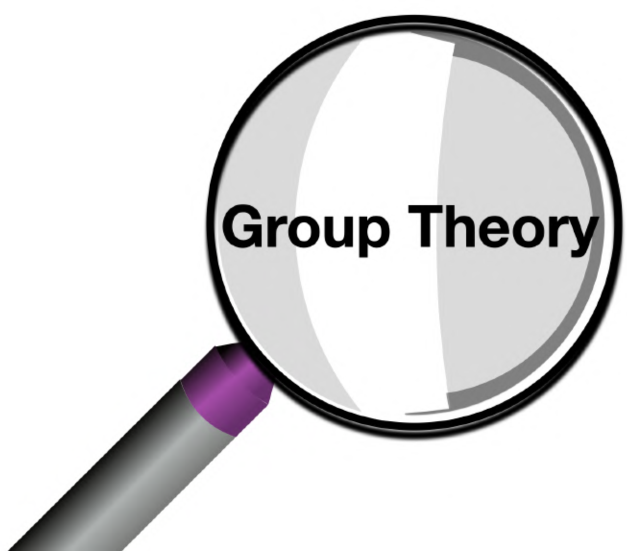
$$V = \left| \vec{a} \cdot \left(\vec{b} \times \vec{c} \right) \right|$$



Now onto **Euclidean Geometry (geometry of motions)**:

The Group of Euclidean Motions

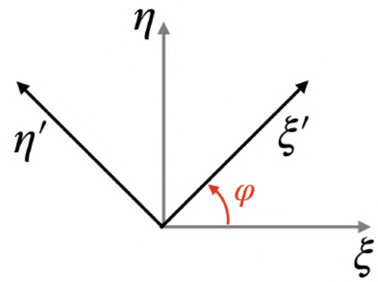
Euclidean geometry can be looked at through the lens of *group theory*.



The symmetries which are fundamental to the Euclidean Space actually form a group of motions: *translations*, *rotations*, and *reflections*, and their compositions, like the rotation in this (ξ, η) plane.

$$\xi' = \xi \cos\varphi + \eta \sin\varphi, \quad \zeta' = \zeta,$$

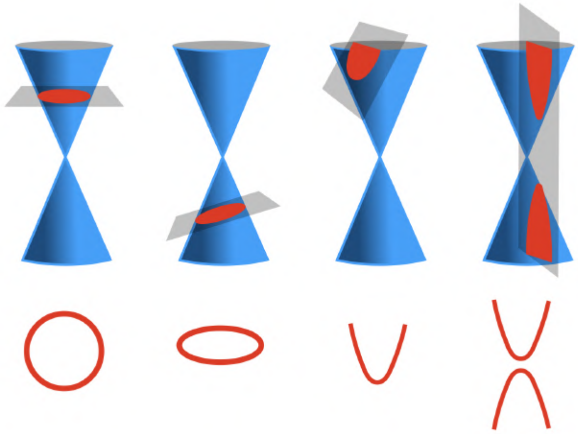
$$\eta' = -\,\xi \sin\varphi + \xi \cos\varphi$$



This way of looking at it, like a *group structure*, holds the idea that Euclidean geometry studies properties which are invariant under rigid transformations.

Conic Sections

Conic sections are curves that are formed by intersecting a plane with a *double cone*. More rigorously, conic sections are “geometric loci which are defined by a constant ratio of distances to a point (otherwise known as *focus*) and a line (or *directrix*)”.



Geometric loci which are defined by a **constant ratio of distances to a point** (otherwise known as focus) and a **line** (or directrix)

Each conic can be expressed in a quadratic equation in Cartesian coordinates

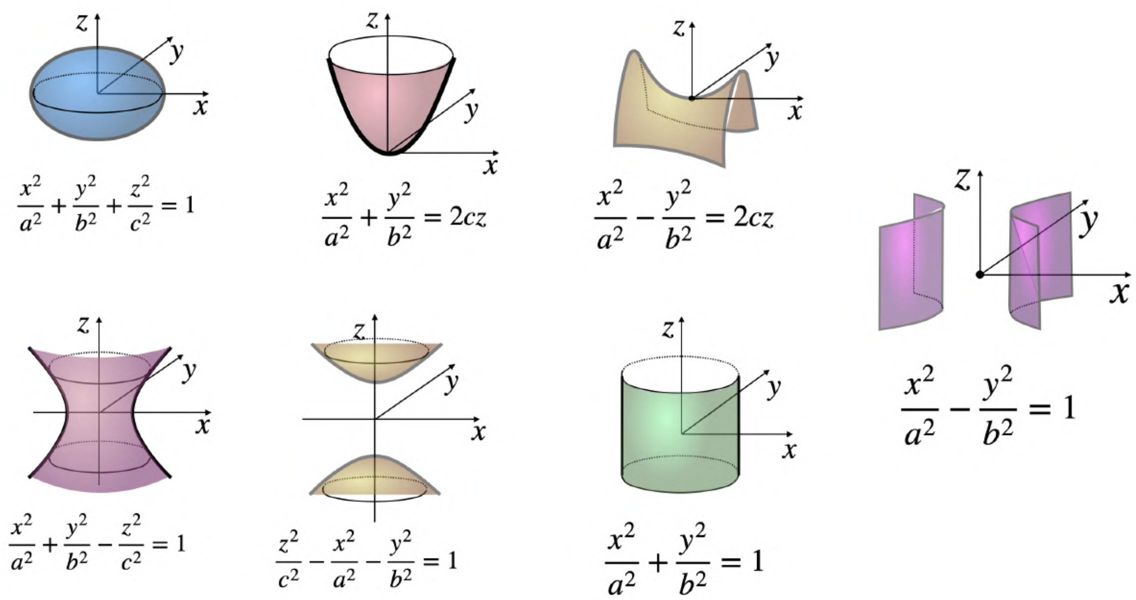
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

We can observe conics in *orbits of planetary motion*, or in *cross-sections of reflective surfaces*, or even as *solutions to second-order differential equations*.

Quadratic Surfaces

These are the $3D$ analogs of conic sections. They are defined by *second-degree equations in* 3 *variables*:

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx + Gx + Hy + Iz + J = 0$$



Depending on what the coefficients are, this equation can represent different types of surfaces, such as *ellipsoids*, *hyperboloids* of one or two sheets, *paraboloids* (either elliptic or hyperbolic), as well as *cones* and *cylinders*, just as some examples.

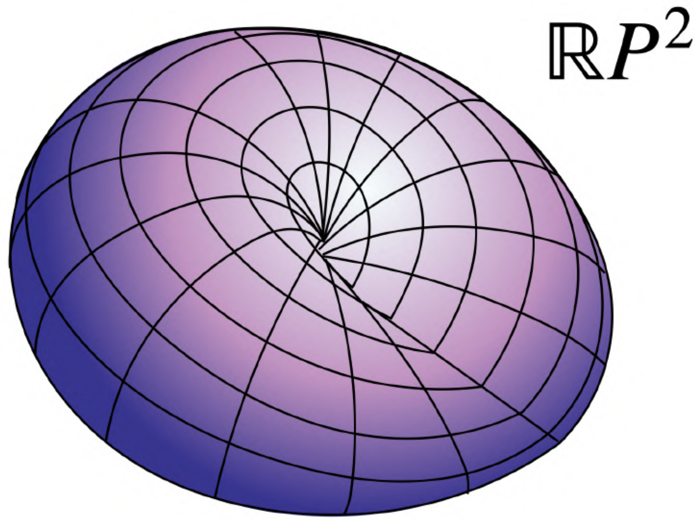
Projective Geometry

Projective Geometry (Basics)

Basically speaking, it studies properties that remain invariant under projective transformations.

a point at infinity = a non-oriented direction

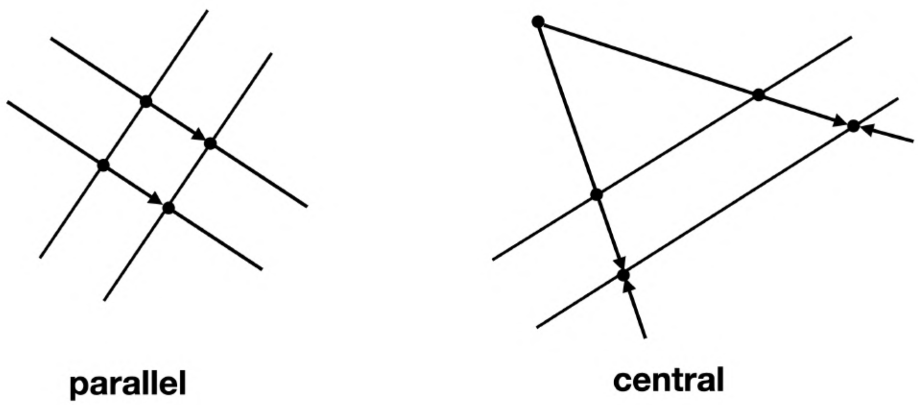
In Euclidean geometry, parallel lines never meet, but in **projective geometry**, all lines intersect, and parallels meet at a *point at infinity*.



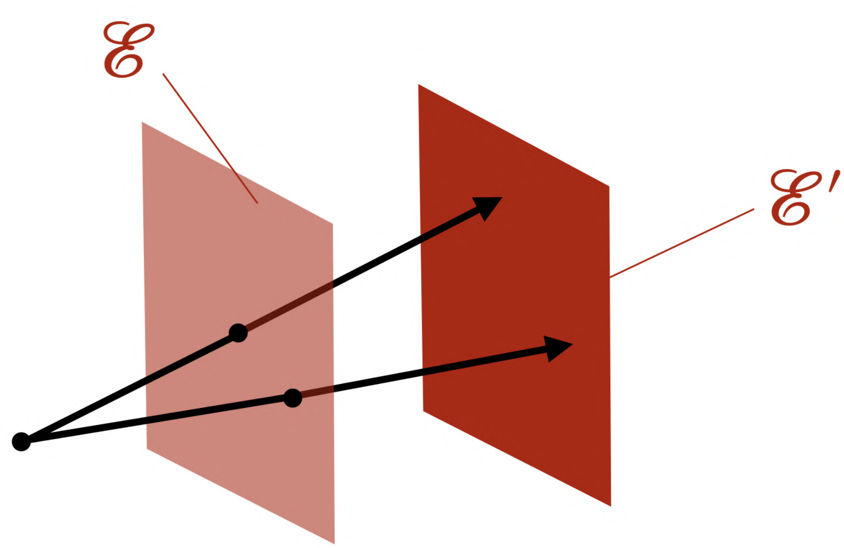
Its basic setting is the *projective plane*, where points are represented using *homogeneous coordinates*, and the lines defined by linear equations in these coordinates. It deals with principles of *duality* and *projective invariants*.

Projective Maps

These are transformations that preserve *straight lines* and *cross ratios* in projective spaces, which are represented by *invertible linear transformations on homogeneous coordinates*.



Basically, under a projective map, distances and angles are not preserved, but properties like *incidence* (so, in simpler words, which points lie on which line) are preserved.

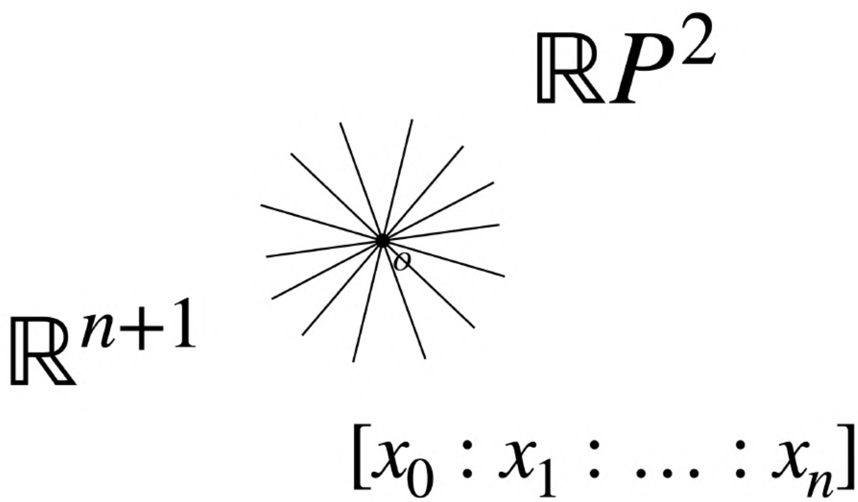


These mappings are represented by 3×3 matrices.

$$\begin{pmatrix} x' \\ y' \\ u' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ u \end{pmatrix}$$

The n-Dimensional Real Projective Space

The **n-Dimensional Real Projective space** is formed by taking all the lines through the origin in $\mathbb{R}^{n+1}$. Each point is represented by homogeneous coordinates $[x_0 : x_1 : \ldots : x_n]$.



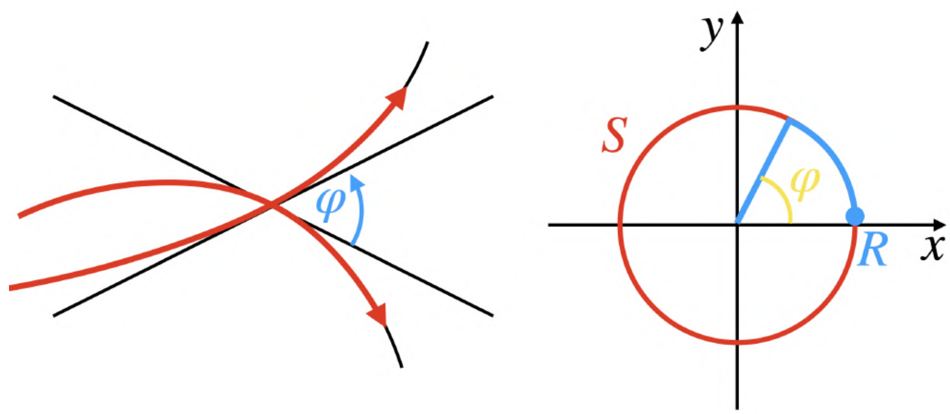
The n-Dimensional Complex Projective Space

The **complex projective space** $\mathbb{CP}^n$ is defined the same way as its real counterpart, as the set of all complex lines through the origin in $\mathbb{C}^{n+1}$, with points represented by homogeneous coordinates $[z_0 : z_1 : \ldots : z_n]$.

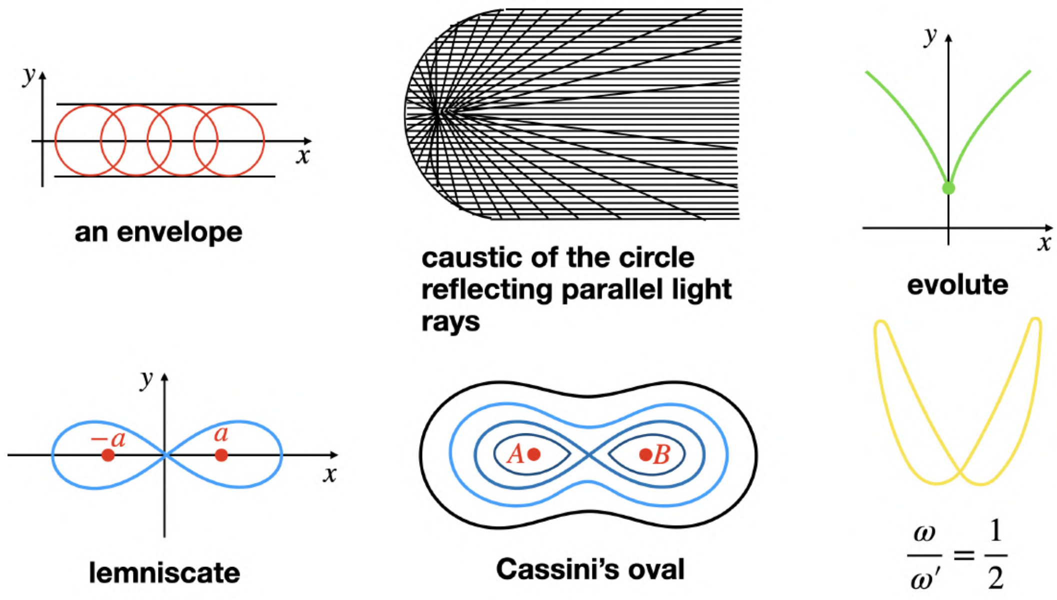
$$\mathbb{C}^{n+1}$$

$$[z_0 : z_1 : \ldots : z_n]$$

Plane Curves

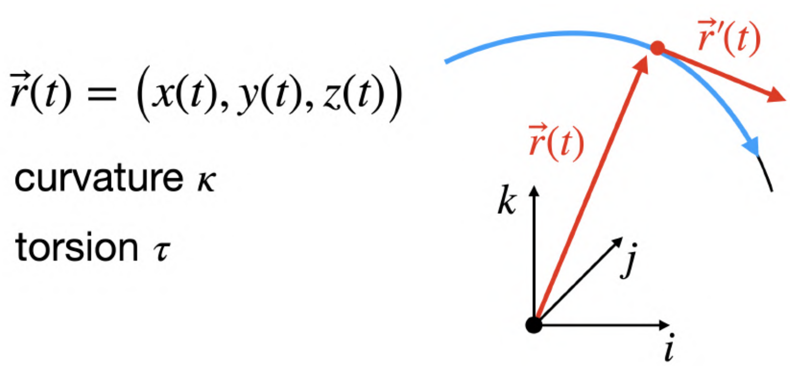


These are 2-*dimensional smooth curves in a plane*, defined by *smooth equations* or *parametrizations* in the plane. They're studied for their geometric features, like *curvature*, *inflection points*, *envelopes*, *symmetries*, and so on.

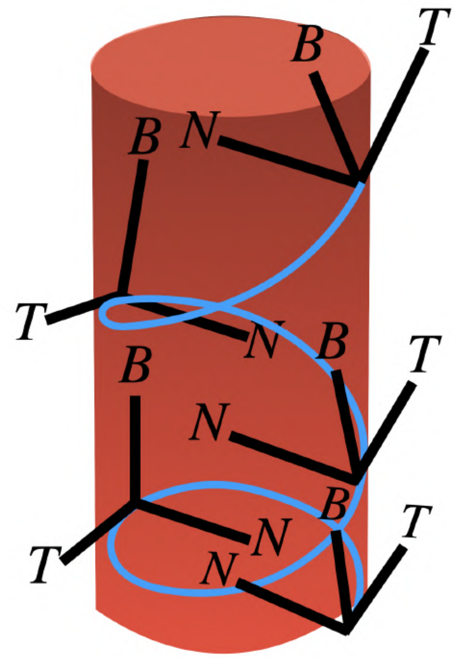


Some examples include *envelopes* and *caustics*, *evolutes*, *lemniscates*, *cassini's oval*, *lissajou figures*, and many, many more.

Space Curves



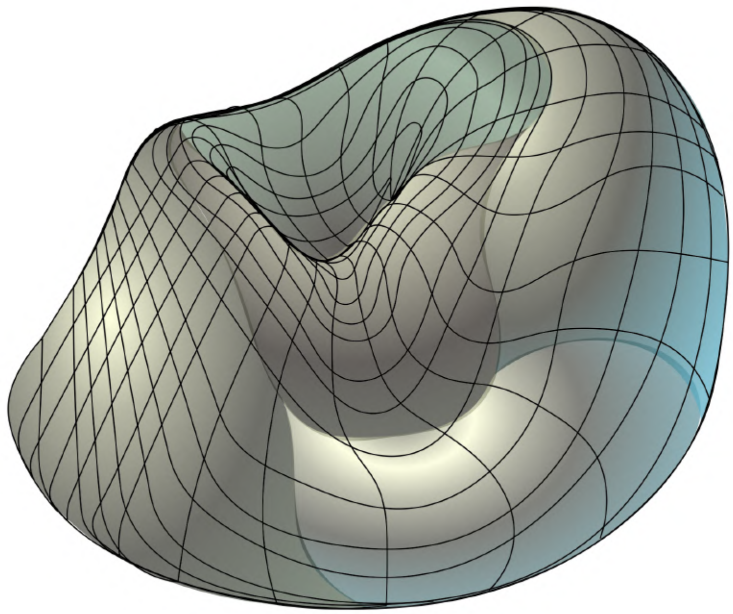
Space curves are studied as *smooth vector-valued functions* $\vec{r}(t) = (\, x(t)\, ,\, y(t)\, ,\, z(t)\,)$ in 3-dimensional space. Their geometric behavior is described using curvature κ and torsion τ, which measure how the curve bends and twists in space.



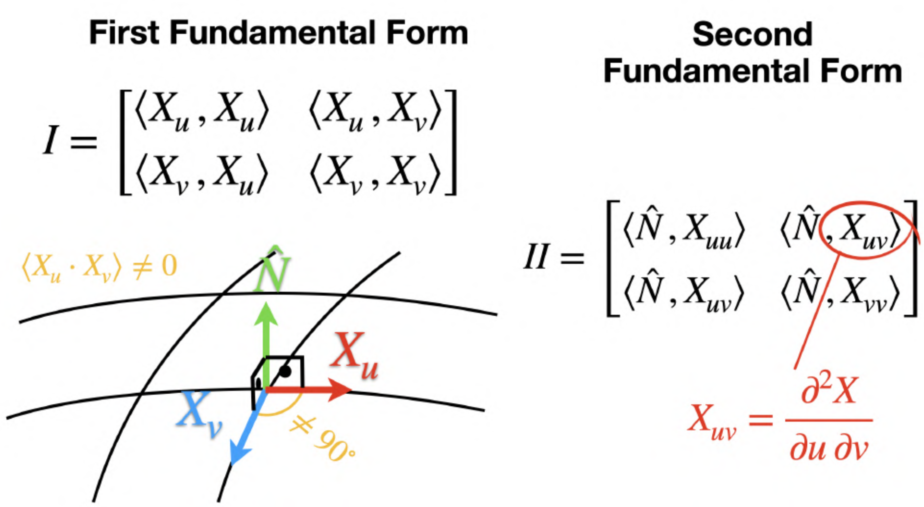
These quantities are captured by the *Frenet–Serret frame*, which are 3 orthonormal vectors: the *tangent* $\vec{T}$, *normal* $\vec{N}$, and *binormal* $\vec{B}$, that move along the curve.

The Gaussian Local Theory of Surfaces

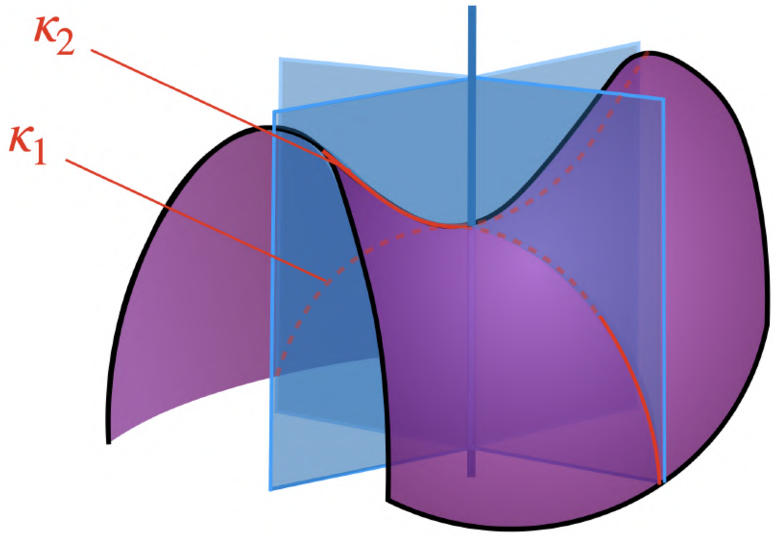
This area studies the intrinsic geometry of *smooth surfaces* in a 3-dimensional space. Particularly, with a focus on how *surfaces curve at each point*.



The geometry of a surface is described by the *first and second fundamental forms*, which hold the metric and curvature properties.

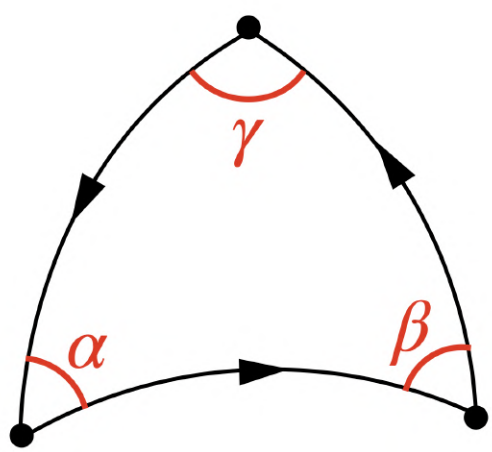


The central idea is *Gaussian Curvature*, denoted by K, which is defined as the product of the *principal curvatures* κ_1 and κ_2.



The amazing thing it shows is that the curvature is an *intrinsic property* of the surface itself, and it's not dependent on how it sits in space.

Gauss' Global Theory of Surfaces



Gauss' global theory connects *local curvature* with the *global topology* of a surface.

Gauss–Bonnet Theorem

$$\int_S K\,dA = 2\pi\chi(S)$$

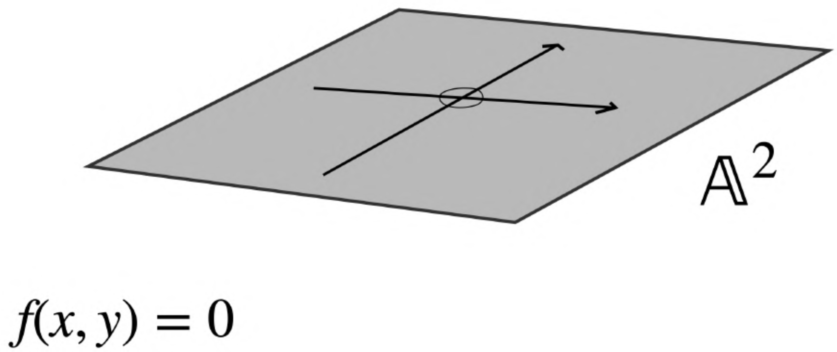
The main result is the *Gauss–Bonnet theorem*. It states that the integral of the Gaussian curvature over a *compact surface S* is related to its *Euler characteristic* $\chi(S)$:

$$\int_S K\,dA = 2\pi\chi(S)$$

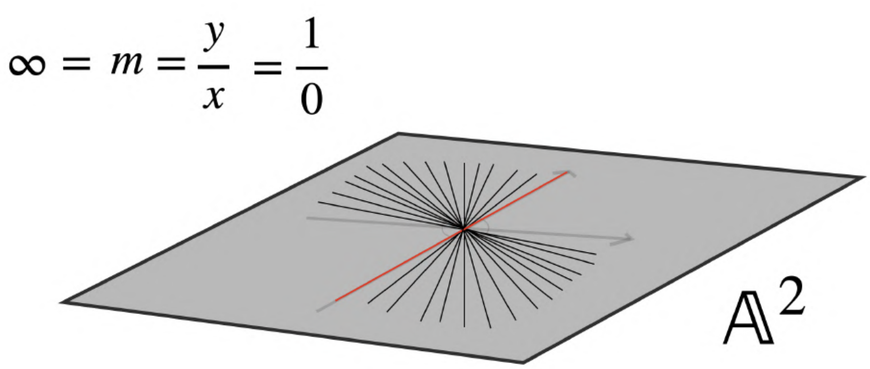
Algebraic Geometry

Algebraic geometry overall is a very very large field, with countless applications in other areas of mathematics. Its key concepts include the *degree of a curve*, *singular points*, *complex numbers in projective spaces* (like trying to avoid missing points at infinity).

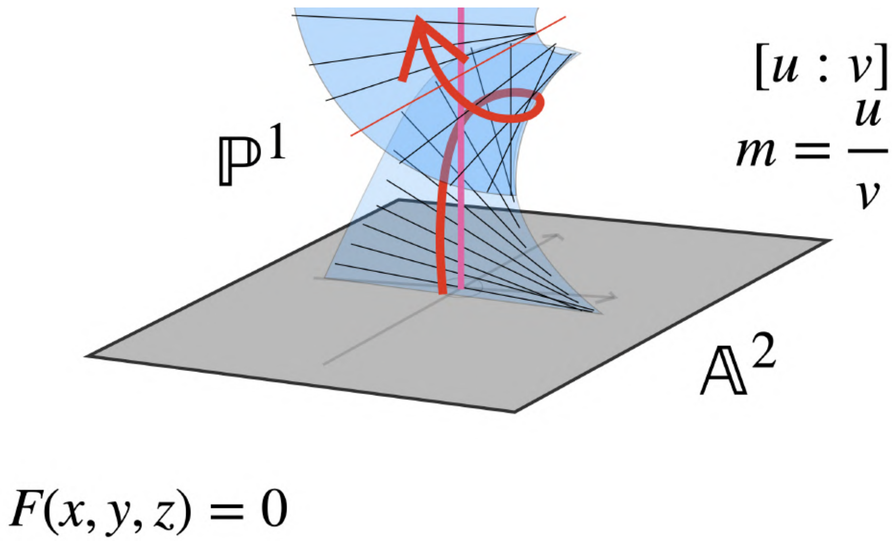
The Projective Complex Form of a Plane Algebraic Curve



This concept explains how to extend a plane algebraic curve to projective space over the complex numbers, which would form a projective complex algebraic curve.



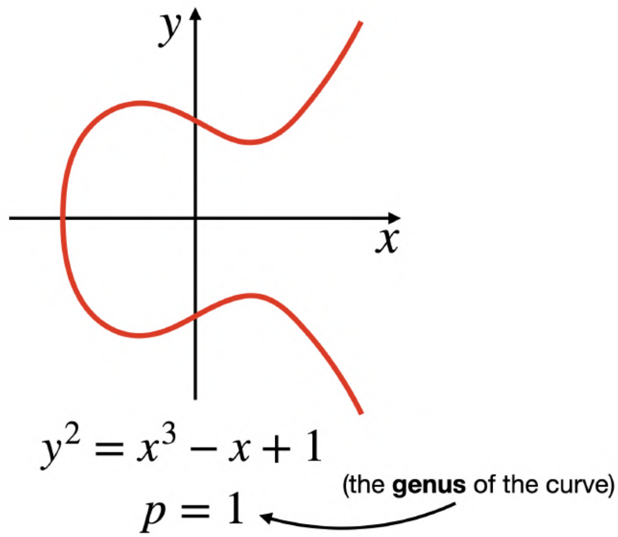
This is done by introducing homogeneous coordinates and rewriting the equation as a *homogeneous polynomial* $F(x, y, z) = 0$, which makes sure that points "at infinity" are included.



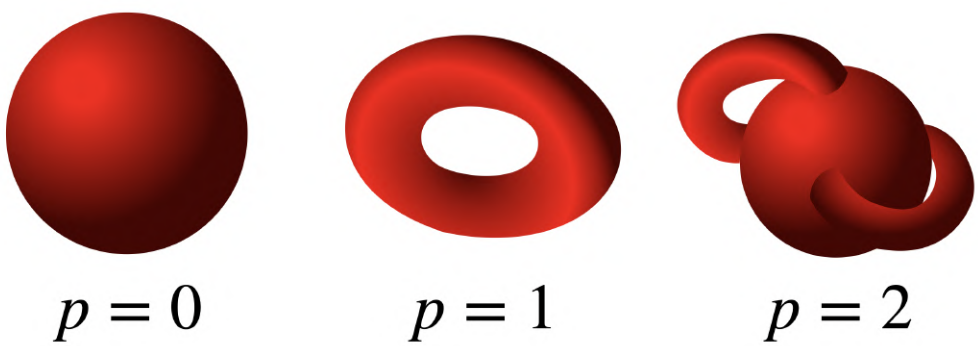
We have to be working in a *complex projective space* though $\mathbb{CP}^2$, because this way we make sure that every algebraic curve becomes *compact* and *closed*, thus avoiding the incomplete behavior that is seen in *affine coordinates*.

The Genus of a Curve

Although there are certainly more accurate, but complicated ways of describing it, the **genus of a curve** is a number that tells you how many "holes" or loops the shape of the curve has when viewed as a smooth surface over the complex numbers (for the image below, the genus is $p = 1$).



For example, a sphere has genus 0, a torus has genus 1, and so on.



Diophantine Geometry

Diophantine geometry studies the solutions of polynomial equations with *integer* or *rational number coefficients*. These are called *Diophantine equations*.

$$a = m^2 - n^2 \qquad b = 2mn \qquad c = m^2 + n^2$$

m and n are positive integers

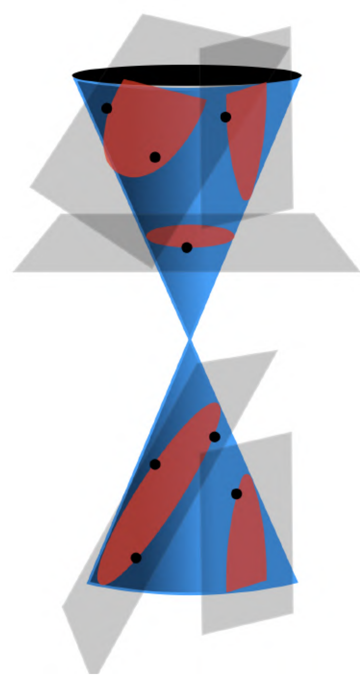
$m > n > 0$

m and n have opposite parity

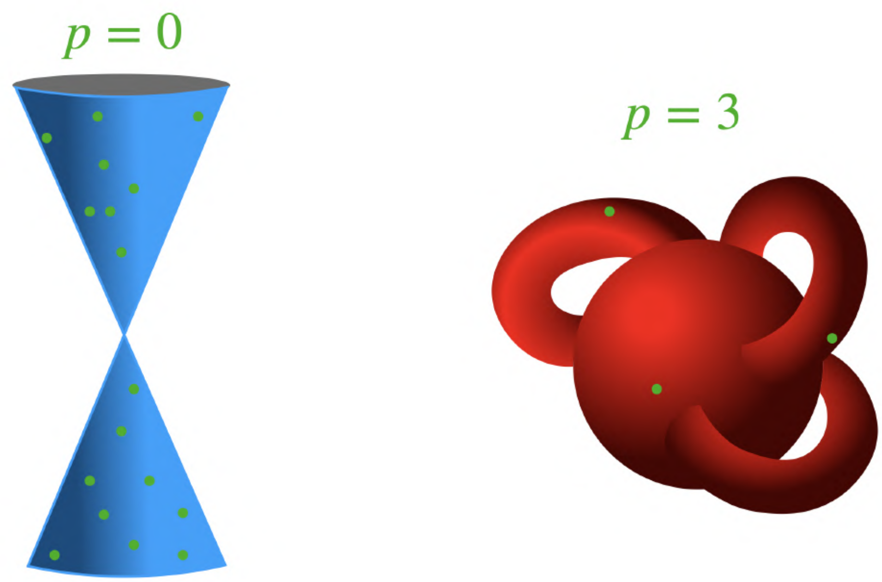
m and n are coprime.

The main idea is to treat the equation as defining a *geometric object*, and then ask whether it contains any *rational points*, as well as how many of them.

$$x^2 + y^2 = 7z^2$$



There's an interesting relationship between the genus of a curve and its rational solutions. For example, curves of genus 0 (like conics) can have infinitely many rational points if they have even one, but curves of genus ≥ 2 have, by *Faltings' theorem*, only finitely many rational points.



Analytic Sets and the Weierstrass Preparation Theorem

Although *analytic sets* can have several definitions, we'll look at them as the solution sets of systems of *holomorphic equations*. These sets can have *singularities*, which means that they are in need of local analysis to understand their structure.

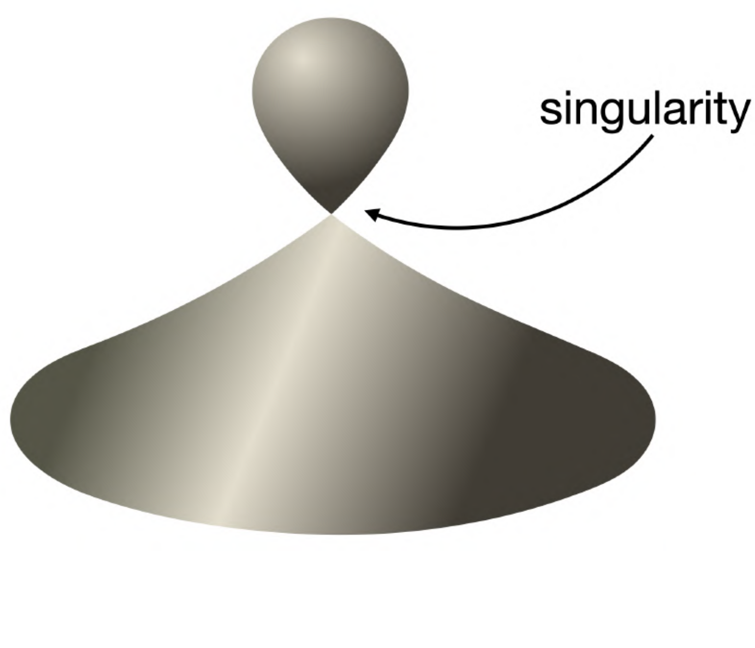
The Weierstrass Preparation Theorem

Let $f: U \subseteq \mathbb{C}^n \to \mathbb{C}$ be **holomorphic** in a **neighborhood** of the origin,such that the function $w = f(0, t)$ is a **power series** beginning with the term t^k. Then there is a uniquely determined **factorization** of the form $f(z, t) = p(z, t)g(z, t), \quad g(0,0) \neq 0$

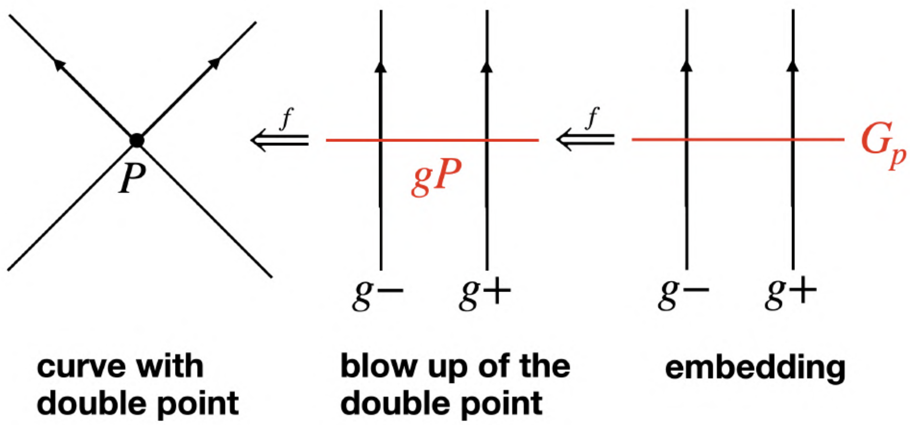
The key tool is the *Weierstrass Preparation Theorem*, which gives a way to simplify holomorphic functions near a singularity. It's the local analytic tool, which doesn't resolve the singularity itself, but is a broader step.

The Resolution of Singularities

It's pretty straightforward, resolving singularities means transforming a space with singular points into one that is *smooth*, while preserving its essential *geometric* or *topological structure*.



For example, when a curve has a singular point, like a place where it crosses itself, that point can be "blown up". Instead of treating it as a single point, we replace it with an entire line that captures all the directions in which the curve approaches. This process makes it easier to study the curve near that point.



There are many methods to resolve singularities, but *Hironaka's theorem* guarantees that the resolution of singularities is always possible, but only in characteristic 0.

The Algebraization of Modern Algebraic Geometry

What we mean by that are *schemes*, introduced by Alexander Grothendieck, which generalize *varieties* by allowing local models given by *rings*, not just coordinate functions.

The fundamental notion of a **scheme:**

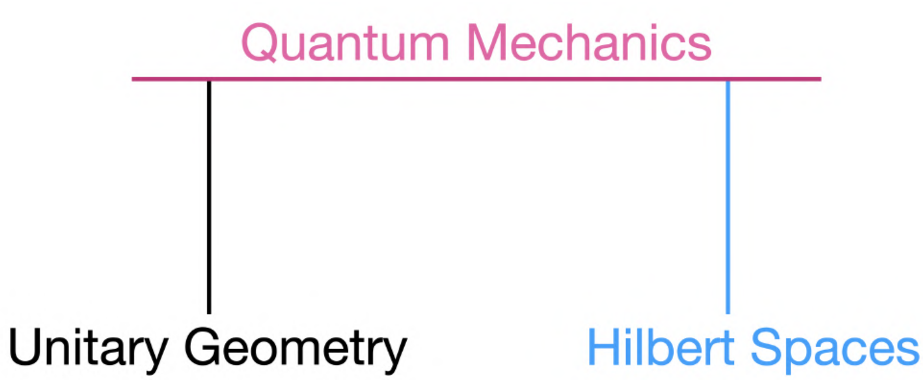
a **ringed space** $(X, \mathscr{G})$

is said to be **scheme** if it looks locally like a given ringed space $(X_0, \mathscr{G}_0)$

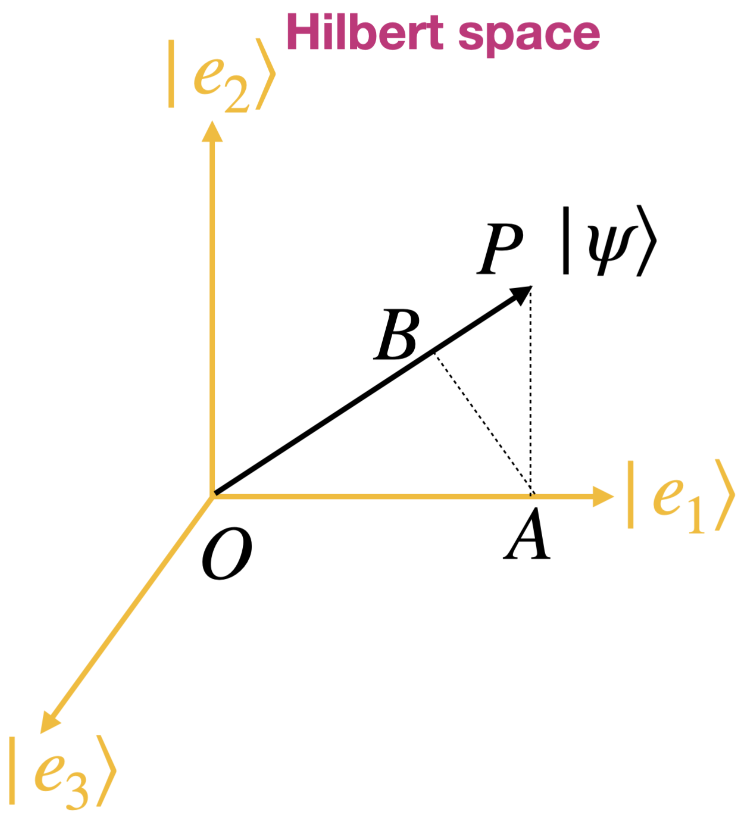
Geometries of Modern Physics

Geometries that go beyond Euclidean geometry. Things like *manifolds*, *tensor calculus*, and *differential geometry*, which are used to model *physical space*, *spacetime*, and *fields*.

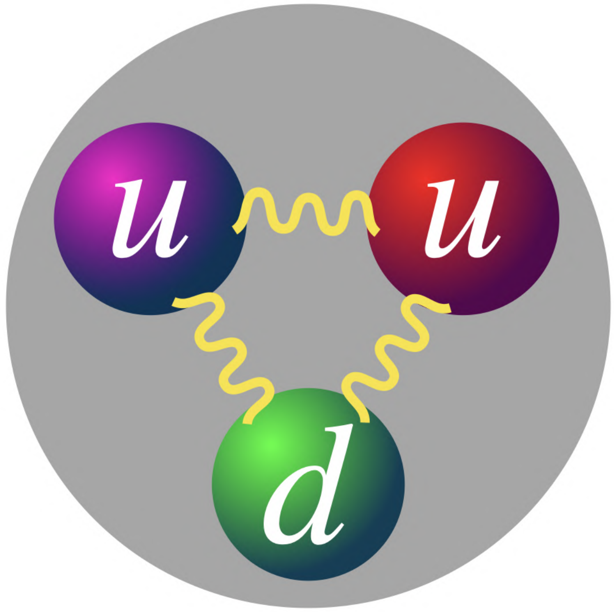
Unitary Geometry, Hilbert Spaces, and Elementary Particles



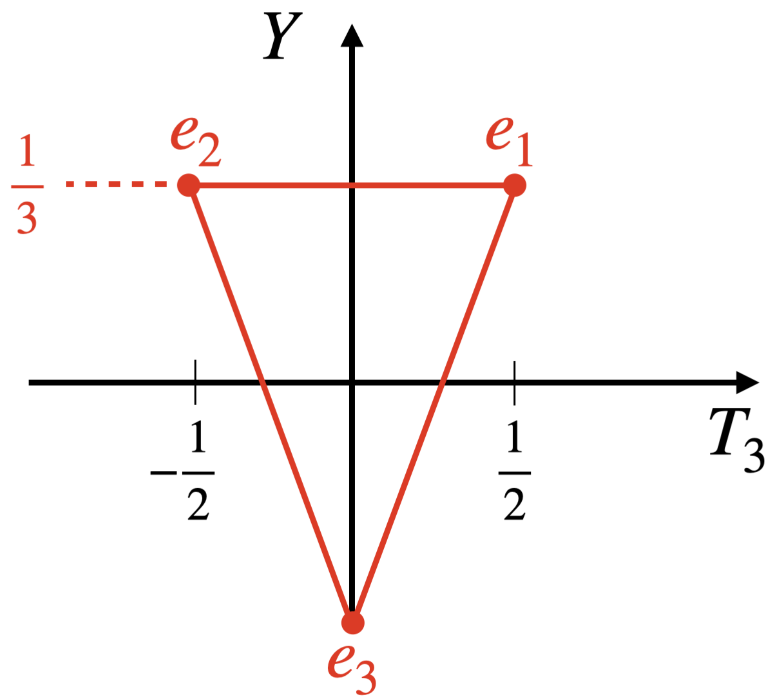
Unitary geometry and Hilbert spaces form the mathematical foundation of *quantum mechanics*.



A *Hilbert space* is an ∞-dimensional complex vector space that is equipped with an *inner product*. It lets us precisely formulate *states*, *observables*, and *evolution* in quantum theory, for example.



Quarks are an example of how *unitary geometry* and Hilbert spaces describe *elementary particles*. Each *quark flavor* corresponds to a vector in a Hilbert space, and their charges, like *isospin* and *hypercharge*, determine how they transform under *unitary symmetry groups*.



Pseudo-Unitary Geometry

Pseudo-unitary geometry is not a real geometry that we experience with our everyday world, but it is the geometry of Einstein's *special theory of relativity*.
For example, if we use the real vector space $\mathbb{R}^2$ with a *bilinear form* defined as:

$$B(u,v) := u_1 v_1 - u_2 v_2$$

then, the *pseudo-orthogonal group* $O(1,1)$, preserving the bilinear form, is given by matrices involving *hyperbolic functions* $\cosh(\alpha)$ and $\sinh(\alpha)$. These describe "boosts" like in special relativity.

$$B(u, v) := u_1 v_1 - u_2 v_2$$

$$O(1,1)$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = (\pm 1) \begin{pmatrix} \cosh \alpha & -\sinh \alpha \\ -\sinh \alpha & \cosh \alpha \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \quad \alpha \in \mathbb{R} .$$

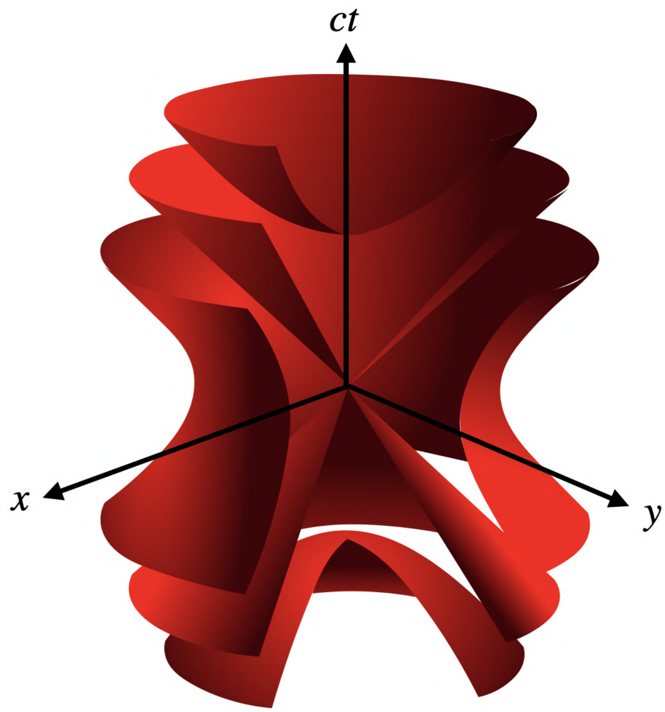
$u \neq 0$ for which $B(u, u) = 0$ $\qquad u = (1,1)^T$

isotropic

Vectors $u \neq 0$ for which $B(u, u) = 0$, like $u = (1, 1)^T$, are called *isotropic*. They exist only in indefinite geometries, not in Euclidean or unitary spaces.
In this space, the two axes behave oppositely under the inner product. This creates a geometry where "isotropic" directions exist, which means non-zero vectors with zero length.
Since this bilinear form has signature $(1, 1)$, it means that one direction contributes positively, the other negatively. This makes it a *pseudo-unitary space of Morse index* 1.

Minkowski Geometry

Minkowski geometry is the geometric framework of special relativity, where space and time are unified into a 4-dimensional spacetime with an *indefinite metric*.



$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

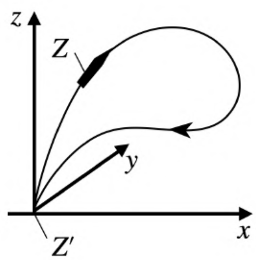
The key object is the Minkowski space. It's a vector space equipped with the *Lorentzian inner product*

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2$$

Applications to the Special Theory of Relativity

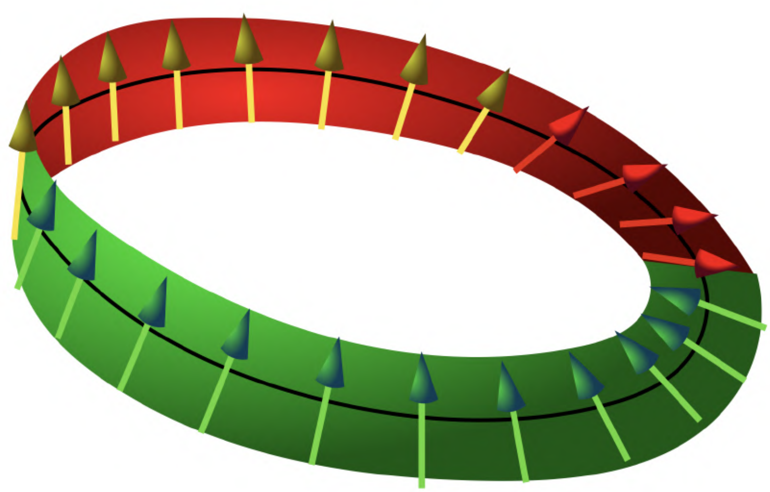
This includes areas like *Lorentz Transformations*, the *eigentime*, *Einstein's twin paradox*, the *Maxwell Equations of electrodynamics*, and so on.

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \mathscr{A} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} + \begin{pmatrix} x_0 \\ y_0 \\ z_0 \\ ct_0 \end{pmatrix} \qquad \tau = \int_{t_0}^{t} \sqrt{1 - \frac{\mathbf{x}'(\sigma)^2}{c^2}}\, \mathrm{d}\sigma$$

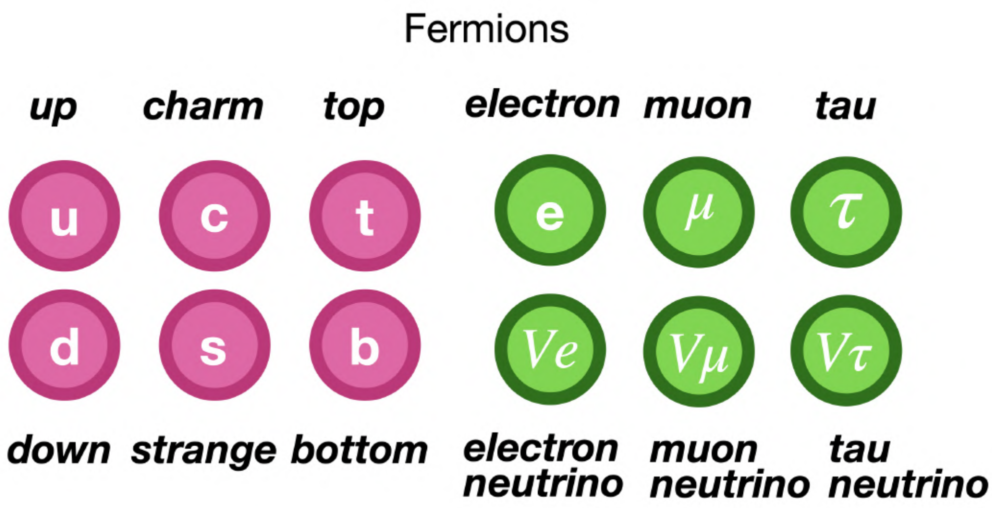


$$\mathrm{Div}\, F = J, \ \mathrm{Div} * F = 0$$

Spin Geometry and Fermions



Spin geometry extends differential geometry to include *spinor fields*, which are needed if we want to describe *fermions*.

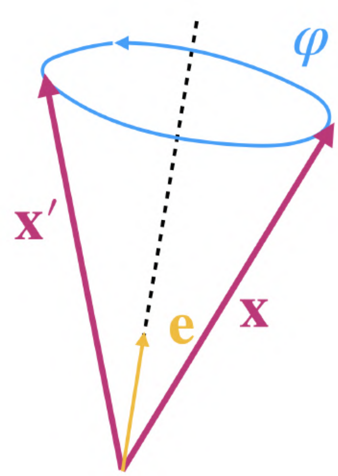


Fermions are particles like quarks and electrons that obey something called the *Pauli exclusion principle*.

This framework is necessary for writing down the *Dirac equation*, which is the *relativistic wave equation* for spin-$\frac{1}{2}$ particles.

For example, rotations in space can be described using *quaternions*, which is a concept from Hamilton that encodes a rotation as a product involving a special object Q, constructed from the axis of rotation and the angle.

$$x' = Q \vee x \vee \overline{Q}$$

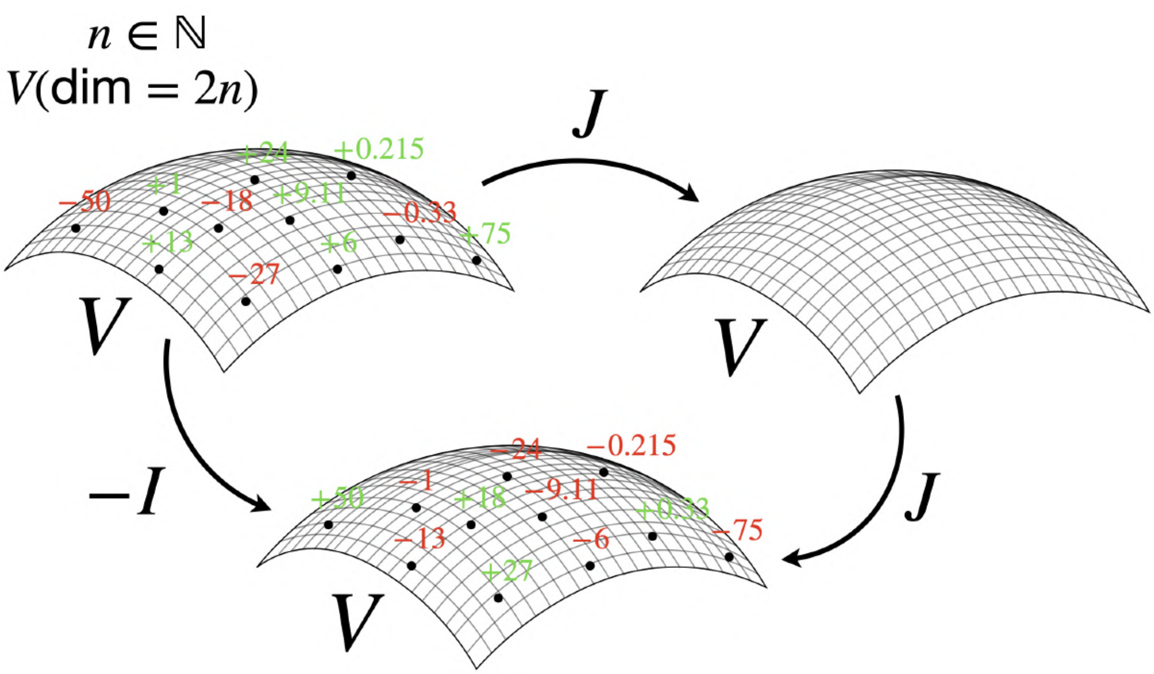


The formula $x' = Q \vee x \vee \overline{Q}$ expresses how a vector x is rotated around an axis e by an angle φ. The goal here is to show how this compact expression, which works great in 3 dimensions, can be extended to higher dimensions using *Clifford algebras*.

This sets up the bigger idea in spin geometry, which is that spinors and fermions naturally live in this more general algebraic framework, where rotations, geometry, and quantum behavior are unified.

Almost Complex Structures

An **almost complex structure** is a way of giving a real space of *even dimension* a rule that behaves like multiplication by the imaginary unit i. This rule is a *linear map J* that, when applied twice, gives *minus the identity*.



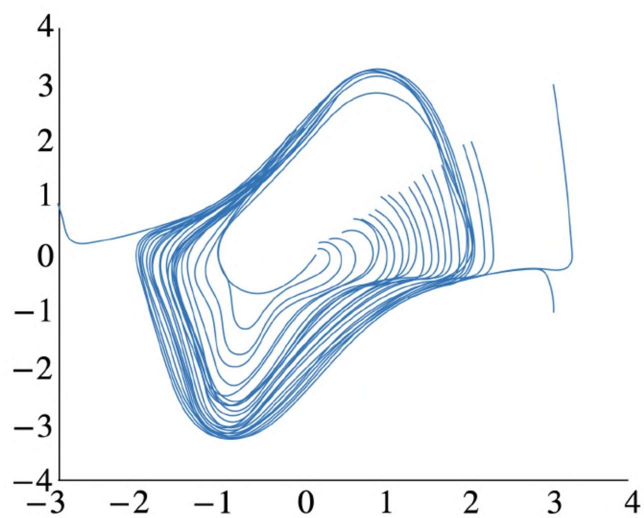
It lets us treat the real space as if it had *complex structure*, even if it doesn't come from actual complex coordinates.

Symplectic Geometry

Symplectic geometry studies even-dimensional manifolds that are equipped with a *closed*, *non-degenerate 2-form* ω, called a *symplectic form*. ω holds the *phase space* relations between positions and momenta.

$$(i)\ \omega \text{ is } skew-symmetric, \quad \omega(u,v) = -\,\omega(v,u)\ \forall\ u,v, \in X$$

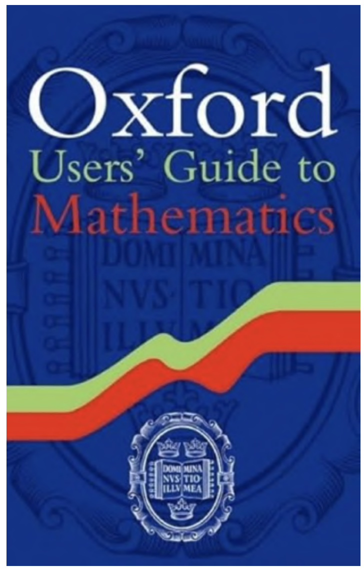
$$(ii)\ \omega \text{ is } non-degenerate, \text{ from } \omega(u,v) = 0\ \forall\ u \in X, \text{ then } v = 0$$



Unlike Riemannian geometry, symplectic geometry lacks things like distance or angles but is rich in *dynamical structures*.

We of course can't possibly show every single thing in geometry, but these are some of the most important concepts. If we missed something let us know in the comment section below.

This file is completely based on the *Oxford Users' Guide to Mathematics*:



Oxford Users' Guide to Mathematics.

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