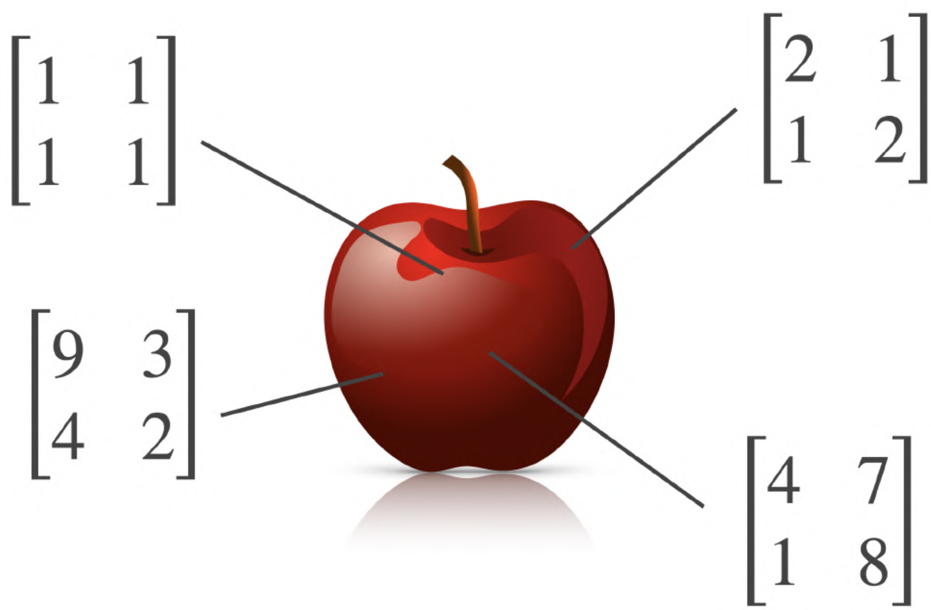
What Is a Metric?

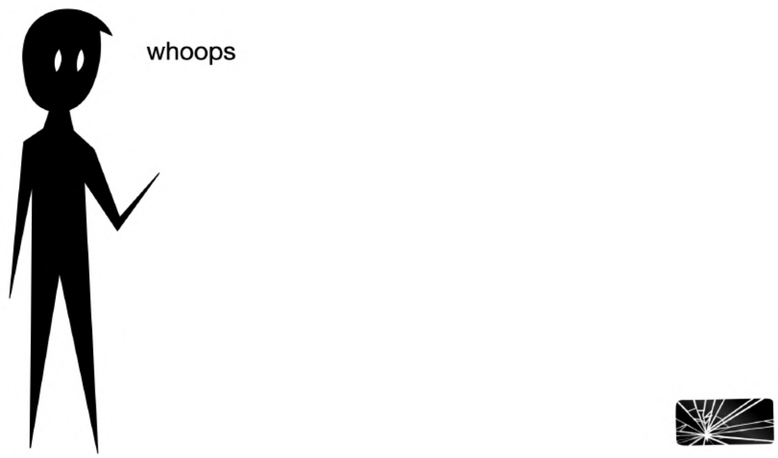
by DiBeos



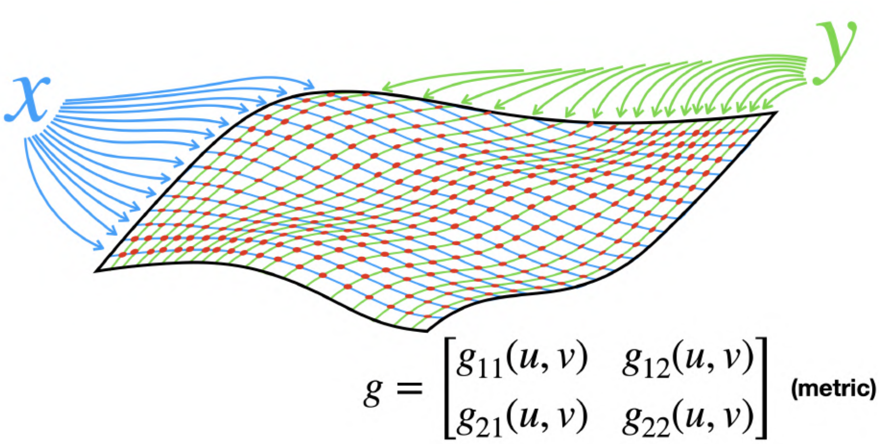
Contents

1 Introduction

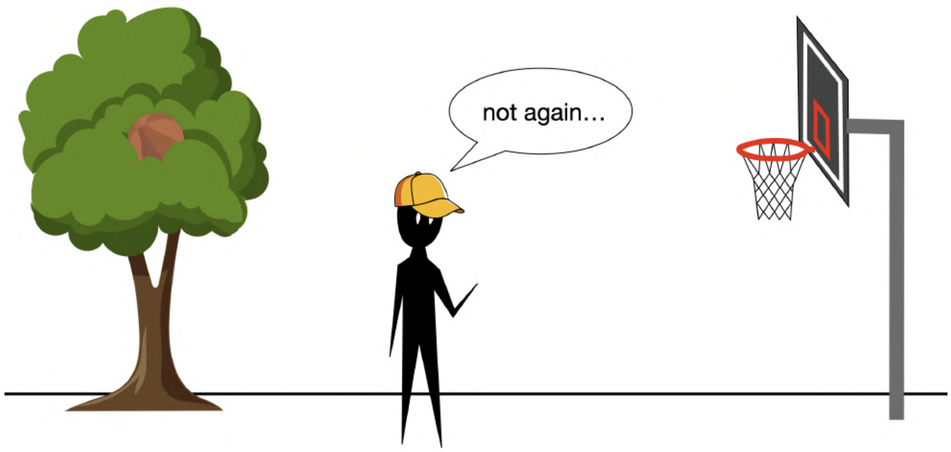
We are surrounded everyday by curved objects – roads, bridges, hills, and apples... But did you know that there is a deeper structure to reality most people never think about? This structure dictates how objects move, from the path of a falling iphone to the macroscale of the Universe.



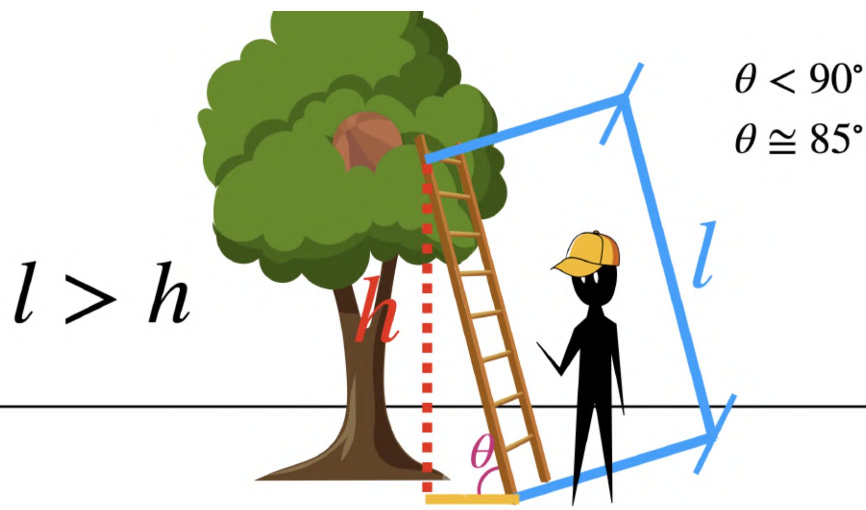
Beneath every curve in a space... there's a hidden formula ruling it all. It's called the *metric*. And understanding it is like gaining access to the operating system of the Universe.



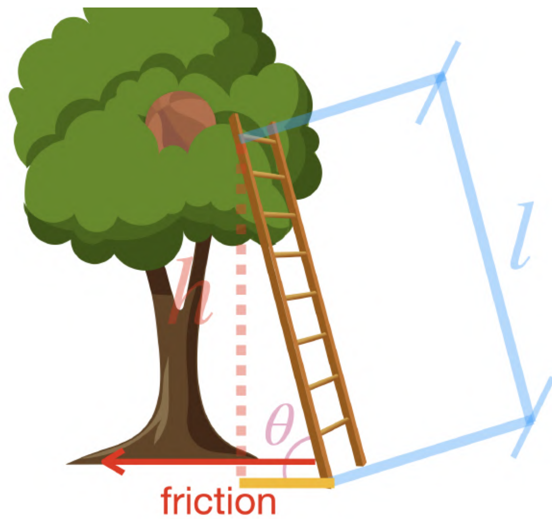
2 Intuition



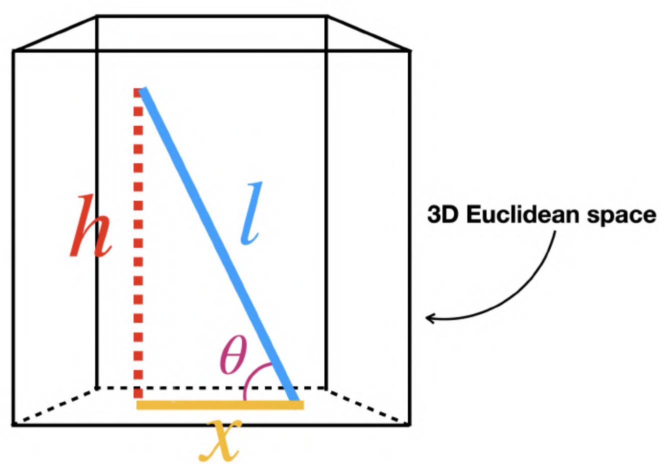
Imagine you're a kid playing basketball in your backyard. The ball bounces out of bounds and gets stuck in a tree. Classic. So you grab a ladder to get it down.



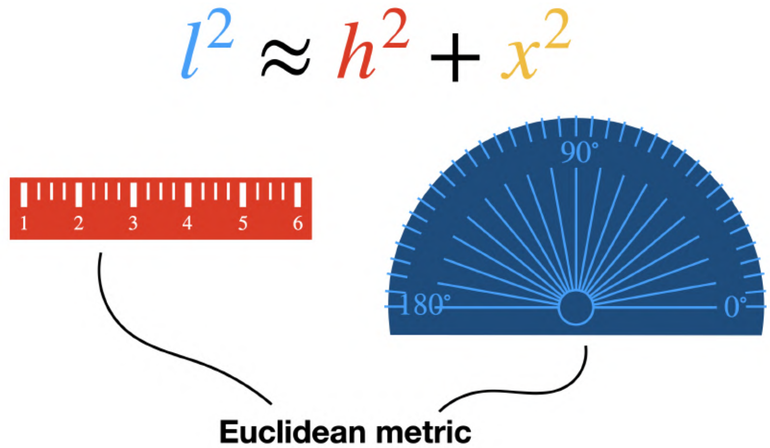
Of course, you would need to adjust its height so that it is greater than the distance between the ground and where the ball is located on the tree. This means that you would need to create a right triangle such that the angle θ formed between the ground and the ladder is close to $90°$ (let's say $\theta \approx 85°$?), otherwise the friction force would not be strong enough to hold you still.



Anyway, that's not the point here. The point is that all of these things, so *lengths* and *angles*, can be easily measured once you assume that we live in a Euclidean flat 3-dimensional space. Which is a pretty good approximation.



What we often fail to notice though is that when performing these calculations we are also assuming a specific ruler or a specific protractor as the only available option. This is the standard *Euclidean metric*.



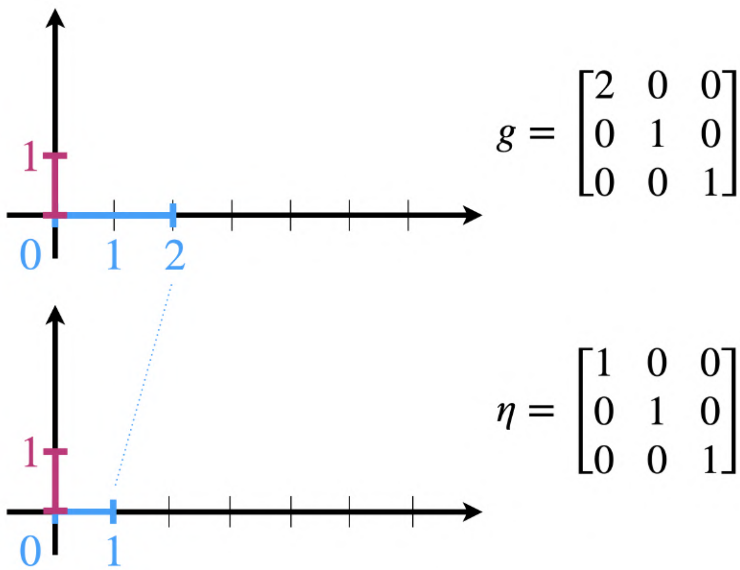
However, there are others... for example the matrix

$$g = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

is a metric in flat space, but not an Euclidean one because it's not proportional to the identity matrix:

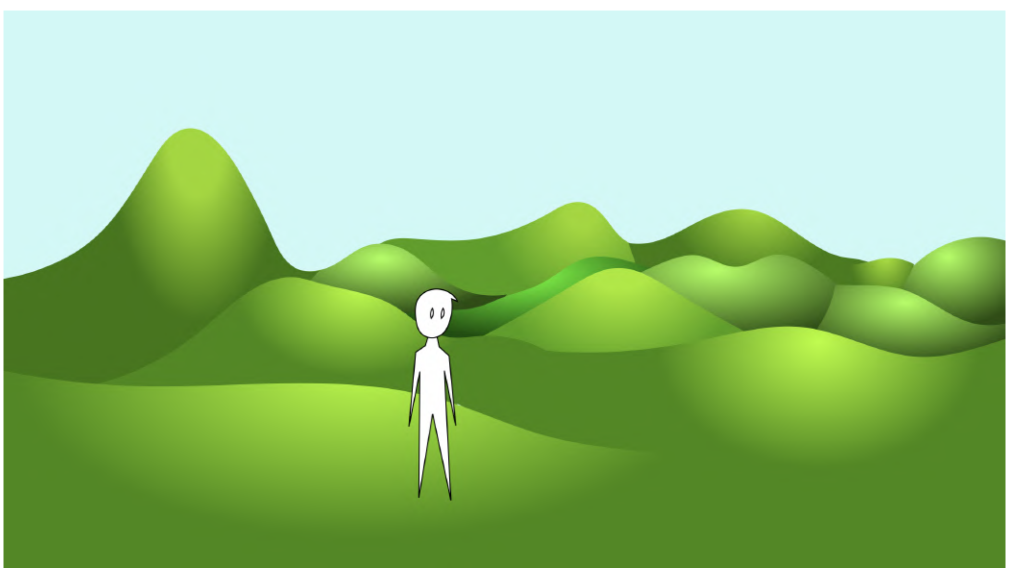
$$\eta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

This metric "stretches" distances in the x-direction compared to the y-direction. It basically says that one unit step in x is twice as long as it would be in the Euclidean metric, but one unit step in y is unchanged.

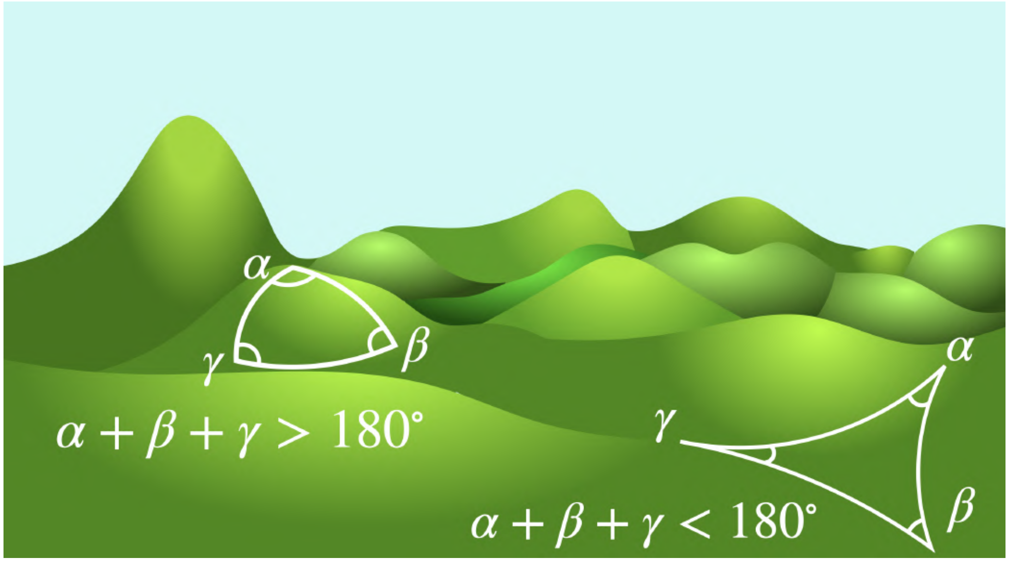


It is definitely a bad physical model of our adjustable-ladder-and-tree illustration, but mathematically it is consistent. So, it really depends on the context. The bottom line is that the metric is just a matter of convenience – a choice.

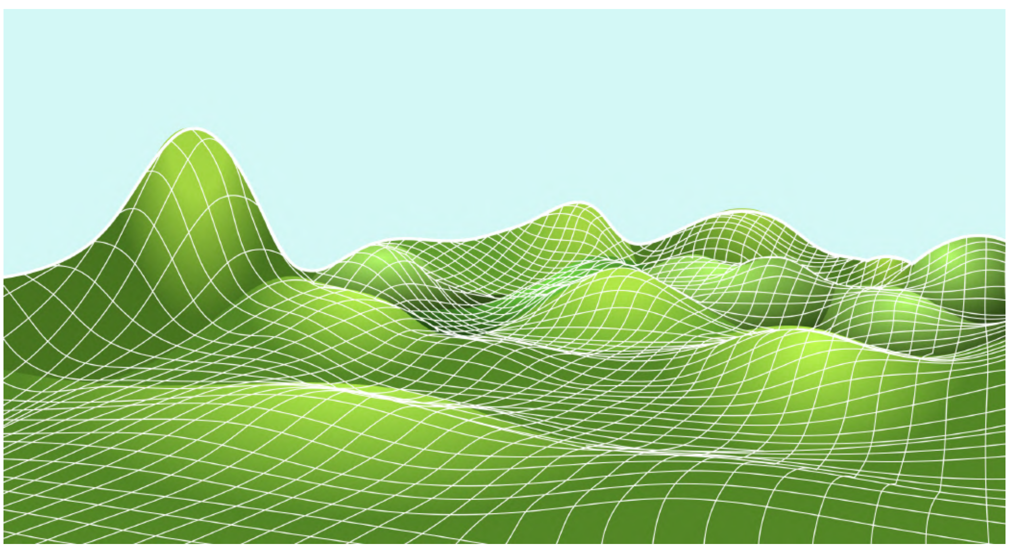
Let's see another illustration. This time though in a curved space.



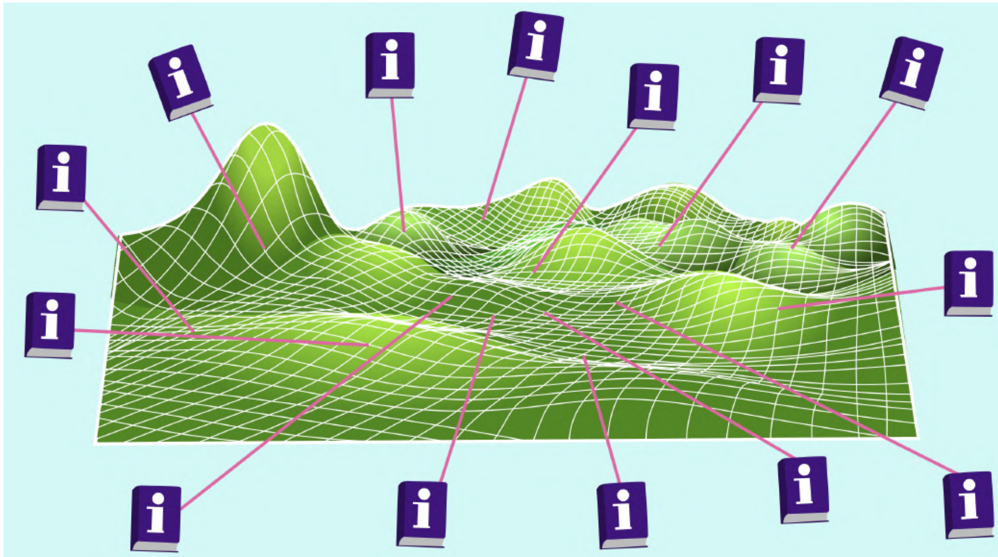
Now imagine you're not in your backyard anymore. Instead, picture yourself in a landscape, with rolling hills. You try to draw a large right triangle on the ground, but something seems off...



You sum up the angles that you measured but they just don't add up to 180° anymore. In some regions the result is greater than 180° and in others less than 180°. Well, the issue is obvious, isn't it? The space you are dealing with is not flat anymore, it is curved. And now, your Euclidean rulers don't work anymore...



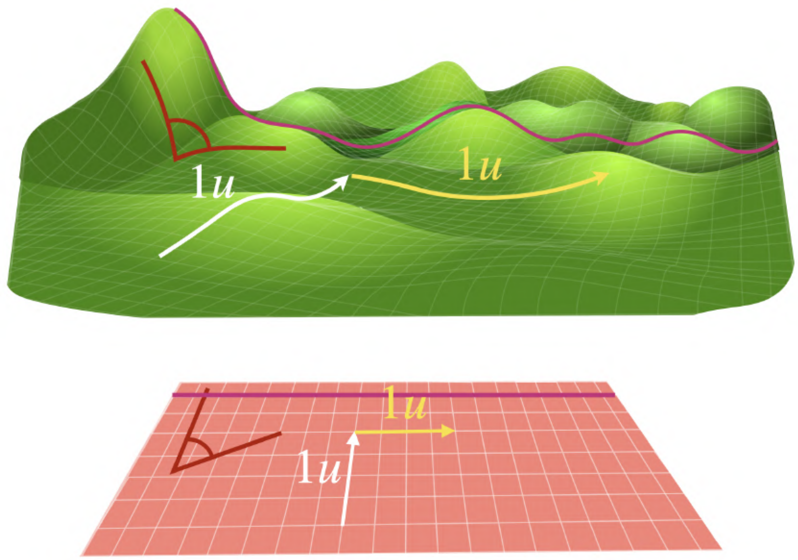
Geometry depends on the metric that defines how things are measured, locally, at every point. Here, you can still draw triangles, but not the Euclidean ones. The rules have changed, and the metric is the new "rule book" now.



At each point in this field, there's a sort of instruction manual, like a tiny blueprint, that tells us how lengths and angles should behave in the immediate neighborhood. This instruction manual is the metric.

So, instead of thinking about the metric as a straight ruler, imagine it's more like a guide, or manual, given to every point of space. It tells you:

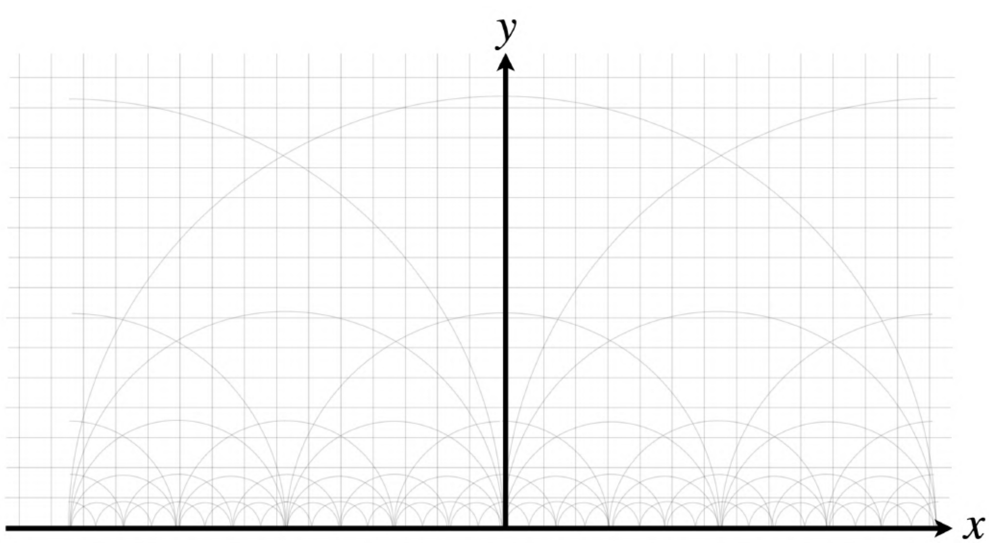
If you want to move just one unit north and one unit east, how much ground will you really cover? If you rotate something, what angle have you swept based on how this region bends? And if you try to go "straight", what does straight even mean here?



The neighborhood of each point will obey its own set of rules (locally), dictated by its own metric, and the way these instructions connect across the surface defines the full geometry (globally). That's what makes the metric so powerful: it's locally defined but its compound influence is global throughout the entire space.

Great! Now that you have the intuitive picture of what a metric is, let's dive deeper with a concrete example.

3 Concrete Example



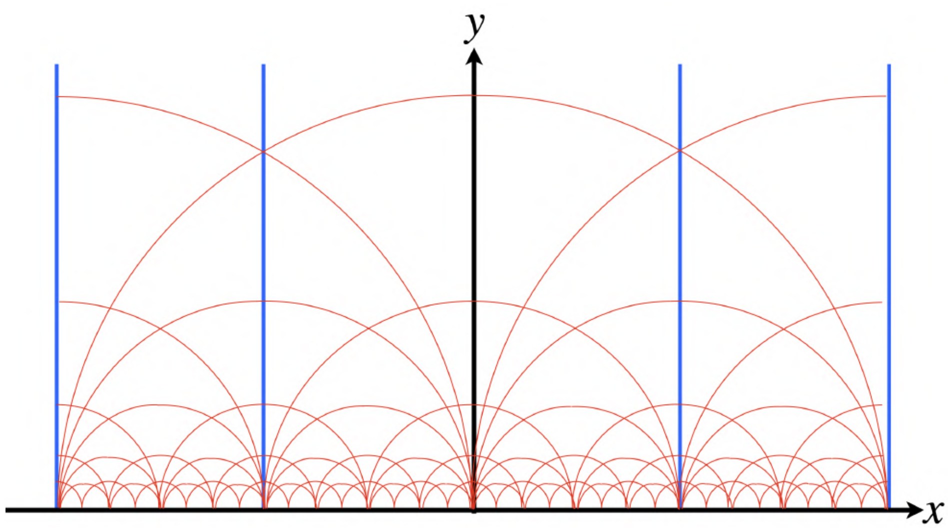
This is a 2-dimensional space. I know, it looks flat, but it is not, because we will define a non-trivial metric in it. This metric will be defined by the matrix:

$$g = \begin{bmatrix} \frac{1}{y^2} & 0 \\ 0 & \frac{1}{y^2} \end{bmatrix}$$

This metric is called the *Poincaré half-plane metric* and can also be expressed this way:

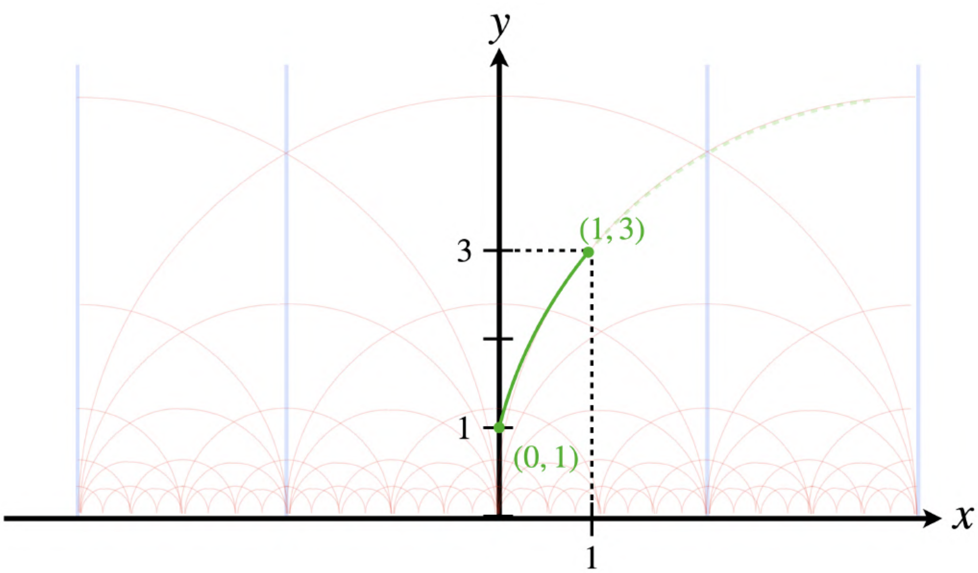
$$ds^2 = \frac{dx^2}{y^2} + \frac{dy^2}{y^2}$$

This will represent only the upper half ($y > 0$). This is a space with constant negative curvature (i.e., Gaussian curvature $K = -1$), and geodesics are either vertical lines or semicircles orthogonal to the x-axis ($\implies$ they diverge).



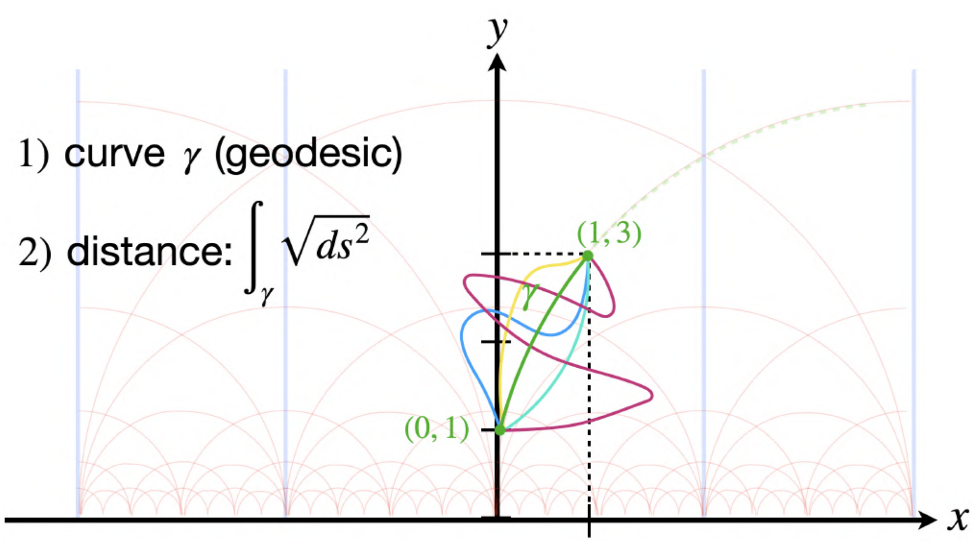
This is a *hyperbolic space*, by the way.

How can we use this metric to measure distances? Let's say we want to measure the distance from the initial point $(x_i, y_i) = (0, 1)$ and the final point $(x_f, y_f) = (1, 3)$.



To compute distances, we need two things:

1. The geodesic curve γ that connects the initial point to the final one.



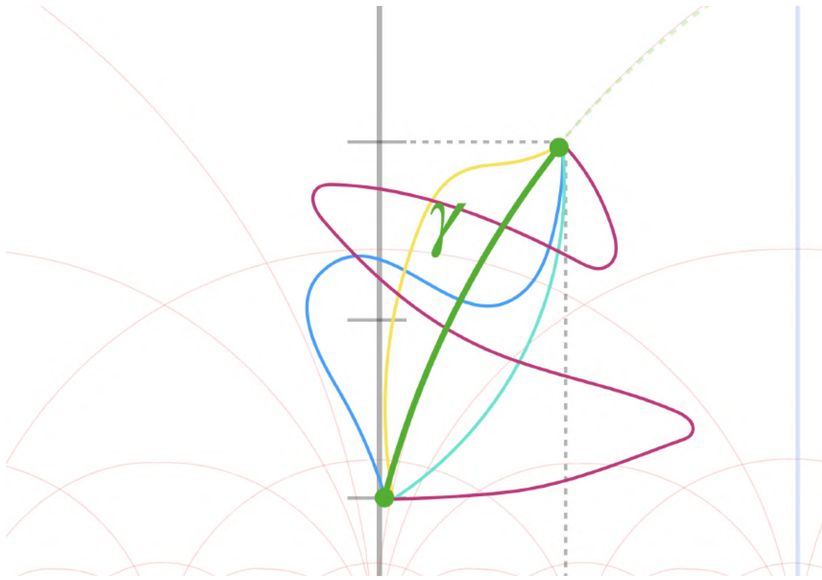
2. Integrate the line element ds along this curve:

$$\underline{Distance}: \quad \int_\gamma \sqrt{ds^2} = \int_\gamma \frac{\sqrt{dx^2+dy^2}}{y} = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{y(x)}\, dx$$

Well, let's calculate it.

So, what we want to find is the curve $\gamma = y(x)$ between two points by minimizing the length functional:

$$\mathcal{L}\left[y\right] = \int_{x_i}^{x_f} \frac{\sqrt{1+\left(\frac{dy}{dx}\right)^2}}{y(x)}\, dx$$



This is a classic problem in *Calculus of Variations*. The integrand (also called the Lagrangian) is

$$\mathcal{F}(y, y') = \frac{\sqrt{1+(y')^2}}{y} \quad , \qquad y' = \frac{dy}{dx}$$

All we have to do is apply the *Euler-Lagrange equation*:

$$\boxed{\frac{d}{dx}\left(\frac{\partial \mathcal{F}}{\partial y'}\right) - \frac{\partial \mathcal{F}}{\partial y} = 0}$$

But, since $\mathcal{F}$ doesn't depend explicitly on x, we reduce the Euler-Lagrange equation to the *Beltrami identity*:

$$\boxed{\mathcal{F} - y'\,\frac{\partial \mathcal{F}}{\partial y'} = C \;\; \text{(constant)}}$$

I know it doesn't seem so, but this is a great simplification of the problem.

Then, we plug in equation (I) in equation (II):

$$\mathcal{F}(y, y') = \frac{\sqrt{1 + (y')^2}}{y} \qquad \text{(I)}$$

$$\mathcal{F} - y' \frac{\partial \mathcal{F}}{\partial y'} = C \qquad \text{(II)}$$

$$\text{(II)} \implies \frac{\sqrt{1 + (y')^2}}{y} - y' \frac{\partial}{\partial y'} \left(\frac{\sqrt{1 + (y')^2}}{y} \right) = C \implies$$

$$\frac{\sqrt{1 + (y')^2}}{y} - y' \cdot \frac{1}{y} \cdot (2y') \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{1 + (y')^2}} = C \implies$$

$$\implies \frac{\sqrt{1 + (y')^2}}{y} - \frac{(y')^2}{y\sqrt{1 + (y')^2}} = C \implies C = \frac{1 + (y')^2 - (y')^2}{y\sqrt{1 + (y')^2}} \implies$$

$$\implies \boxed{C = \frac{1}{y\sqrt{1 + (y')^2}}}$$

We will see shortly what the geometrical interpretation of the constant C is, but for now... try to guess it on your own...

Anyway, we can isolate y' instead, and doing so we can clearly see that this is a *first-order differential equation (ODE)* for $y(x)$:

$$C = \frac{1}{y\sqrt{1 + (y')^2}} \implies y\sqrt{1 + (y')^2} = \frac{1}{C} \implies$$

$$\implies \boxed{y' = \frac{dy}{dx} = \sqrt{\frac{1}{C^2 y^2} - 1}}$$

Well, we can solve it by integrating both sides:

$$\frac{dy}{\sqrt{\frac{1}{C^2y^2}-1}} = dx \implies \frac{dy}{\sqrt{\frac{1-C^2y^2}{C^2y^2}}} = dx \implies \frac{C\,y\,dy}{\sqrt{1-C^2y^2}} = dx \implies$$

$$\implies C\int \frac{y\,dy}{\sqrt{1-C^2y^2}} = \int_{x_0}^{x} d\tilde{x} \implies \left| \begin{array}{c} t := 1-C^2y^2 \\ dt = -2C^2y\,dy \end{array} \right| \implies$$

$$\implies C\int \frac{-dt}{2C^2}\cdot\frac{1}{\sqrt{t}} = x - x_0 \implies -\frac{1}{2C}\int t^{-1/2}\,dt = x - x_0 \implies$$

$$\implies -\frac{1}{2C}\cdot\left(2\sqrt{t}\right) = x - x_0 \implies -\frac{\sqrt{t}}{C} = x - x_0 \implies$$

$$\implies -\frac{\sqrt{1-C^2y^2}}{C} = x - x_0 \implies \sqrt{\frac{1}{C^2}-y^2} = x - x_0 \implies \frac{1}{C^2} - y^2 = (x-x_0)^2 \implies$$

$$\implies \boxed{(x-x_0)^2 + y^2 = R^2}\ ,\ \text{where } R := \frac{1}{C}$$

$\therefore$ Geodesics in the Poincaré half-plane are exactly Euclidean semicircles centered at $(x_0\,,0)$, and vertical lines are special cases in which the center is at infinity ($R \to 0 \iff C \to +\infty$).

Remark:

By the way, in the Beltrami identity, the constant C has a clear geometric meaning and can be found precisely because we fixed the initial and final points $(x_i\,,y_i) = (0,1)$ and $(x_f\,,y_f) = (1,3)$.

Recall that $C = \frac{1}{y\sqrt{1+(y')^2}}$.

It is a *first integral* (constant of motion) of the geodesic equation. It relates to the radius $\left(C = \frac{1}{R}\right)$ of the semicircles that define the geodesic paths. So, each distinct value of C picks out a different geodesic (i.e., semicircle).

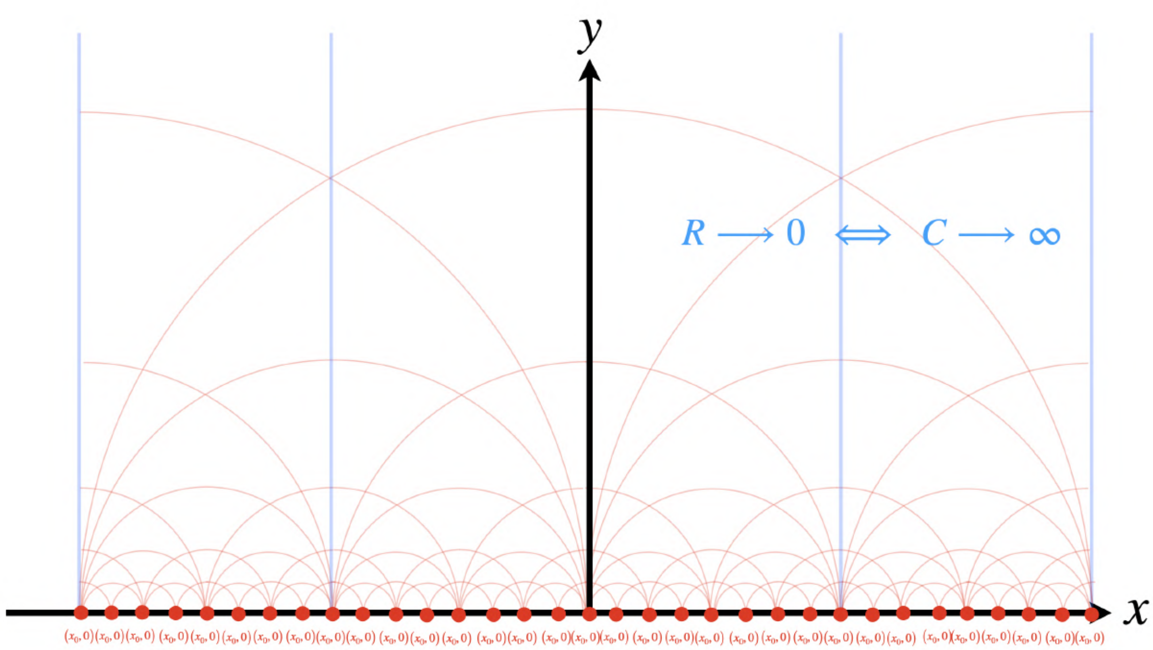
Using the semicircle equation found with the Beltrami identity, we can substitute the initial and final points in order to find the values of x_0 and C:

$$\begin{cases} (x_i - x_0)^2 + y_i^2 = R^2 & (A) \\ (x_f - x_0)^2 + y_f^2 = R^2 & (B) \end{cases}$$

$(B) - (A):$

$$
\begin{aligned}
&(x_i - x_0)^2 + y_i^2 - (x_f - x_0)^2 - y_f^2 = R^2 - R^2 \implies \\
&\implies (0 - x_0)^2 + 1 - (1 - x_0)^2 - 9 = 0 \implies \\
&\implies x_0^2 + 1 - 1 - x_0^2 - 9 = 0 \implies \\
&\implies \boxed{x_0 = \frac{9}{2}} \implies R = \sqrt{(x_i - x_0)^2 + y_i^2} = \sqrt{\left(0 - \frac{9}{2}\right)^2 + 1} = \sqrt{\frac{81}{4} + \frac{4}{4}} \implies \\
&\implies \boxed{R = \frac{\sqrt{85}}{2}} \implies \boxed{C = \frac{2}{\sqrt{85}}}
\end{aligned}
$$

x_0 is the horizontal coordinate of the center of the Euclidean semicircle that represents the geodesic.



Finally, we are ready to calculate the distance between the initial point $(0, 1)$ and the final point $(1, 3)$:

$$
\text{Distance:} \int_\gamma \sqrt{ds^2} = \int_0^1 \frac{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}}{y(x)}\, dx = \int_0^1 \frac{\sqrt{1 + \left(\frac{1}{C^2 y^2} - 1\right)}}{y(x)}\, dx = \int_0^1 \frac{1}{C\, y(x)} \cdot \frac{1}{y(x)}\, dx =
$$

Now we will use the facts that $y^2 = R^2 - (x - x_0)^2$, $x_0 = \frac{9}{2}$, $C = \frac{2}{\sqrt{85}}$ and $R = \frac{\sqrt{85}}{2}$:

$$= \frac{1}{C}\int_0^1 \frac{dx}{y^2(x)} = \frac{\sqrt{85}}{2}\int_0^1 \frac{dx}{R^2-\left(x-\frac{9}{2}\right)^2} = \frac{\sqrt{85}}{2}\int_0^1 \frac{dx}{\frac{85}{4}-\left(x-\frac{9}{2}\right)^2} =$$

$$\left| \begin{array}{c} u := x-\frac{9}{2} \\ du = dx \end{array} \right|$$

$$= \frac{\sqrt{85}}{2}\int_{-\frac{9}{2}}^{1-\frac{9}{2}} \frac{du}{\frac{85}{4}-u^2} = \frac{\sqrt{85}}{2}\int_{-\frac{9}{2}}^{-\frac{7}{2}} \frac{du}{\left(\frac{\sqrt{85}}{2}-u\right)\left(\frac{\sqrt{85}}{2}+u\right)} =$$

$$= \frac{\sqrt{85}}{2}\int_{-\frac{9}{2}}^{-\frac{7}{2}} \left(\frac{A}{\frac{\sqrt{85}}{2}-u}+\frac{B}{\frac{\sqrt{85}}{2}+u}\right) du \qquad (*)$$

Here, A and B are real numbers that must be determined:

$$\frac{A}{\frac{\sqrt{85}}{2}-u}+\frac{B}{\frac{\sqrt{85}}{2}+u} = \frac{1}{\frac{85}{4}-u^2} \implies \frac{A\left(\frac{\sqrt{85}}{2}+u\right)+B\left(\frac{\sqrt{85}}{2}-u\right)}{\frac{85}{4}-u^2} = \frac{1}{\frac{85}{4}-u^2} \implies$$

$$\implies \left(A\cdot\frac{\sqrt{85}}{2}+B\cdot\frac{\sqrt{85}}{2}\right)+(A-B)\,u = 1 \implies$$

$$\implies \begin{cases} \frac{\sqrt{85}}{2}(A+B)=1 \implies A+B=\frac{2}{\sqrt{85}} \implies \boxed{A=\frac{2}{\sqrt{85}}-B} \\ A-B=0 \implies \boxed{A=B} \end{cases}$$

$$\therefore A = B = \frac{1}{\sqrt{85}}$$

Great, now we can proceed with the calculation of the integral $(*)$:

$$(*) \implies \frac{\sqrt{85}}{2}\int_{-\frac{9}{2}}^{-\frac{7}{2}} \left(\frac{A}{\frac{\sqrt{85}}{2}-u}+\frac{B}{\frac{\sqrt{85}}{2}+u}\right) du = \frac{\sqrt{85}}{2}\int_{-\frac{9}{2}}^{-\frac{7}{2}} \left(\frac{\frac{1}{\sqrt{85}}}{\frac{\sqrt{85}}{2}-u}+\frac{\frac{1}{\sqrt{85}}}{\frac{\sqrt{85}}{2}+u}\right) du =$$

$$= \frac{1}{2}\left(-\ln\left|\frac{\sqrt{85}}{2}-u\right|\right)_{-\frac{9}{2}}^{-\frac{7}{2}} + \frac{1}{2}\left(\ln\left|\frac{\sqrt{85}}{2}+u\right|\right)_{-\frac{9}{2}}^{-\frac{7}{2}} =$$

$$= \frac{1}{2}\left(\ln\left|\frac{\frac{\sqrt{85}}{2}+u}{\frac{\sqrt{85}}{2}-u}\right|\right)_{-\frac{9}{2}}^{-\frac{7}{2}} = \frac{1}{2}\left(\ln\left|\frac{\frac{\sqrt{85}}{2}-\frac{7}{2}}{\frac{\sqrt{85}}{2}+\frac{7}{2}}\right| - \ln\left|\frac{\frac{\sqrt{85}}{2}-\frac{9}{2}}{\frac{\sqrt{85}}{2}+\frac{9}{2}}\right|\right) =$$

$$= \frac{1}{2}\ln\left|\frac{\sqrt{85}-7}{\sqrt{85}+7}\cdot\frac{\sqrt{85}+9}{\sqrt{85}-9}\right| =$$

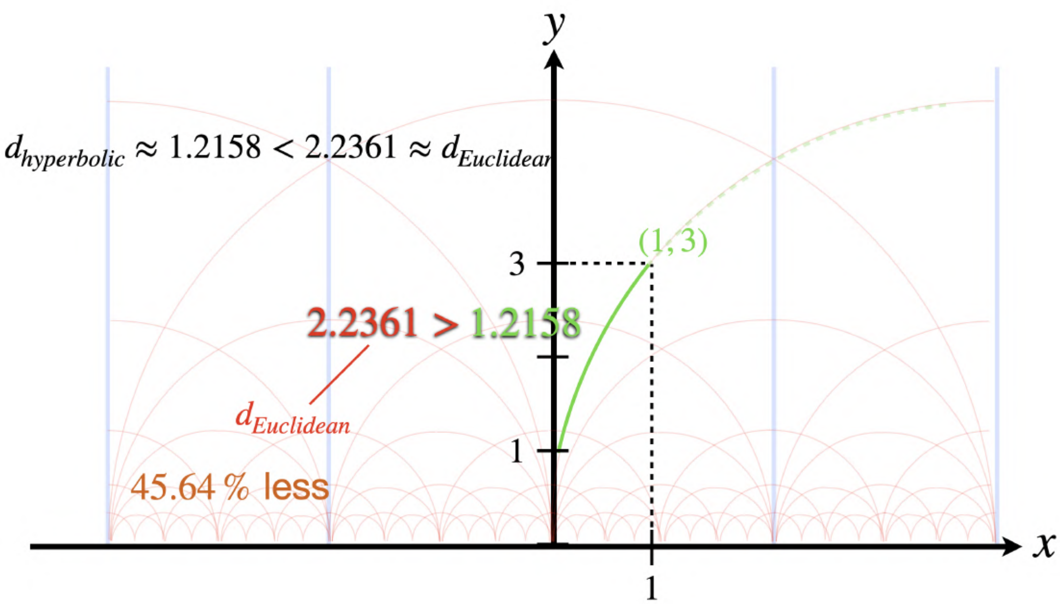
$$= \ln\sqrt{\left|\frac{\left(\sqrt{85}-7\right)\left(\sqrt{85}+9\right)}{\left(\sqrt{85}+7\right)\left(\sqrt{85}-9\right)}\right|} \approx 1.2158$$

Therefore, the shortest path between points $(0,1)$ and $(1,3)$ in this space is 1.2158, which is less than the Euclidean distance using the Pythagorean theorem. Actually, 45.64% less:

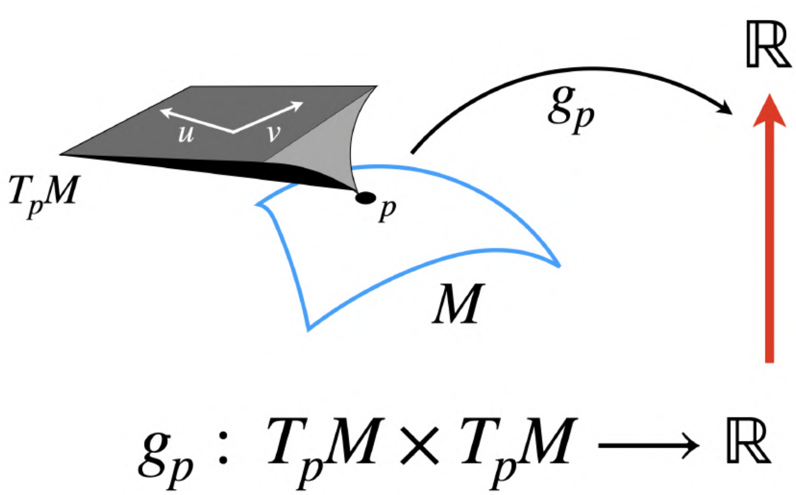
$$d_{Euclidean} = \sqrt{(1-0)^2 + (3-1)^2} = \sqrt{5} \implies \boxed{d_{Euclidean} \approx 2.2361}$$

$$\boxed{d_{hyperbolic} \approx 1.2158}$$

$$\therefore P_{shrinkage} = \left(1 - \frac{d_{hyperbolic}}{d_{Euclidean}}\right) \cdot 100\% = \left(1 - \frac{1.2158}{2.2361}\right) \cdot 100\% \approx 45.64\%$$



4 Rigor



Ok, time to see some of the things we talked about here rigorously:

Let M be a smooth manifold. A metric on M is a function that assigns to each point $p \in M$ a map

$$g_p : T_pM \times T_pM \to \mathbb{R}$$

such that:

1. g_p is *bilinear.*

$\forall a, b \in \mathbb{R}$ and $\forall u_1, u_2, v \in T_pM$:

$$g_p(au_1 + bu_2\,, v) = ag_p(u_1\,, v) + bg_p(u_2\,, v)$$

This is also valid for the second argument:

$$g_p(u\,, av_1 + bv_2) = ag_p(u\,, v_1) + bg_p(u\,, v_2) \quad , \; \forall v_1, v_2, u \in T_pM$$

2. g_p is *symmetric.*

$$g_p(u, v) = g_p(v, u) \quad , \; \forall u, v \in T_pM$$

3. g_p is *non-degenerate.*

$$g_p(u, v) = 0 \;\; , \; \forall v \in T_pM \implies u = 0$$

4. $p \mapsto g_p(u, v)$ is a smooth mapping, $\forall$ smooth vector fields u and v on M.

This means that $g(\cdot, \cdot)$ is *infinitely differentiable*:

$$g\,(X\,, Y) \in C^\infty \quad , \; \forall \text{ smooth } X, Y \in \mathfrak{X}(M)$$

$\mathfrak{X}(M)$ is the set of all vector fields on M.

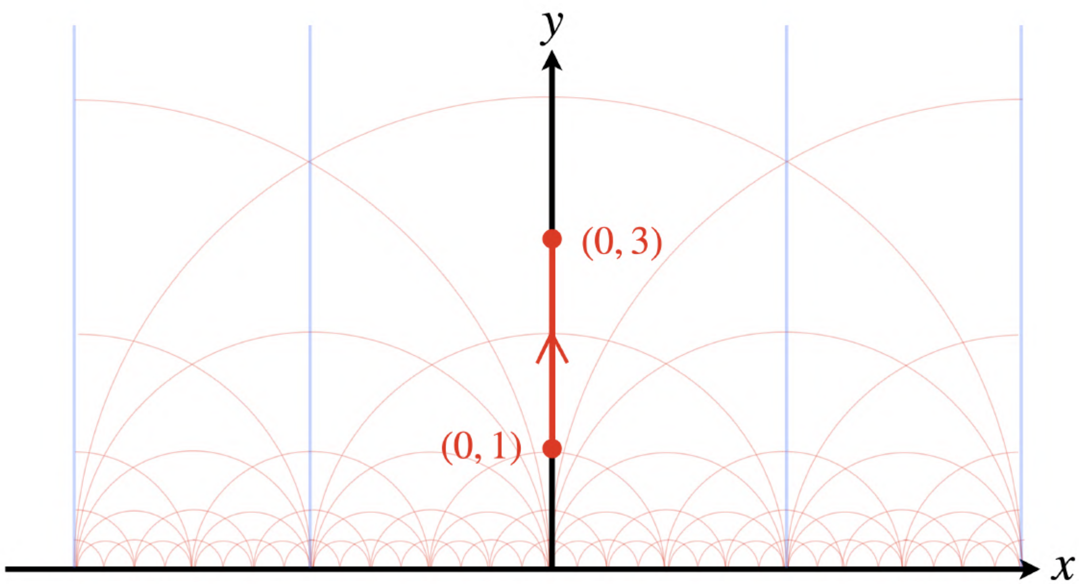
5 Practice

To finish, we will propose an exercise.

Let $\mathbb{H} = \{(x, y) \in \mathbb{R} | y > 0\}$ be the Poincaré half-plane, then it is equipped with the metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}$$

Compute the hyperbolic distance from points $(0, 1)$ and $(0, 3)$ using this metric.



Solution:

Since the geodesic is vertical, $dx = 0$:

$$ds = \frac{\sqrt{dx^2 + dy^2}}{y} = \frac{\sqrt{dy^2}}{y} \implies \boxed{ds = \frac{1}{y}\, dy}$$

$$\underline{Distance} : \int_1^3 \frac{1}{y}\, dy = \ln 3 \quad \approx 1.0986$$

Notice that the Euclidean distance between the points $(0\,,1)$ and $(0\,,3)$ is 2, but in this hyperbolic space we found ≈ 1.0986.

Let's calculate how much *shorter* the hyperbolic vertical distance is relative to the Euclidean one:

$$P_{shrinkage} = \left(1 - \frac{d_{hyperbolic}}{d_{Euclidean}}\right) \cdot 100\% = \left(1 - \frac{1.0986}{2}\right) \cdot 100\% \approx 45.07\%$$

Of course, this percentage is meaningful only for these two specific points. In general, the percentage difference along the vertical y-axis is not constant. In fact, it depends on how far from the origin these two points are. The higher you are (larger y), the smaller a unit Euclidean segment appears in hyperbolic terms. This "rate of shrinkage" is determined by the metric scale $\frac{1}{y^2}$ in the vertical direction.
