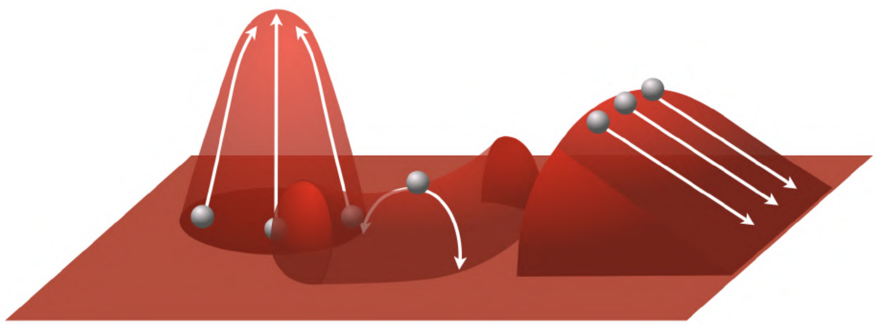
Ricci Curvature

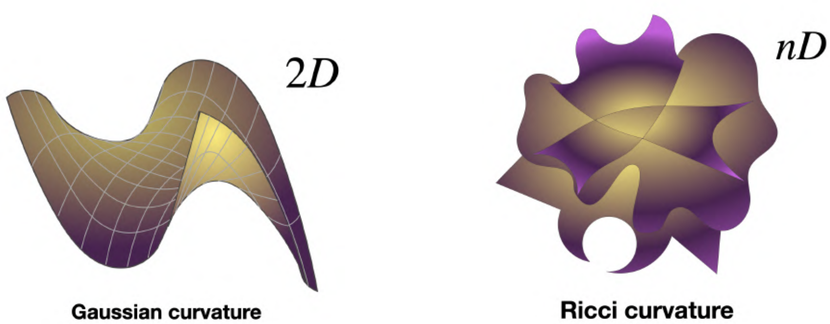
by DiBeos



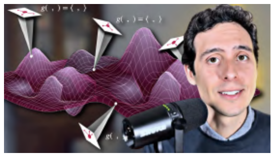
Take a flat space, and place some objects on it. If we were to deform the space, some areas would feel like they're pulling things inward – others push them apart. *Ricci curvature* is the tool that captures this. It tells us how volumes shrink or grow, how paths called geodesics bend together or drift apart.

In physics, it connects mass and energy directly to the shape of spacetime. Ricci curvature is more than math – it's how the universe keeps track of what's inside of it.

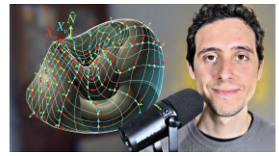
Ricci curvature is very powerful because it applies to any n-dimensional *Riemannian manifold* – not only to 2-dimensional surfaces, like the *Gaussian curvature*.



By the way, if you'd like to know more about these concepts, check out the following videos and PDF links, where we dive deep into these concepts in a VERY clear way:

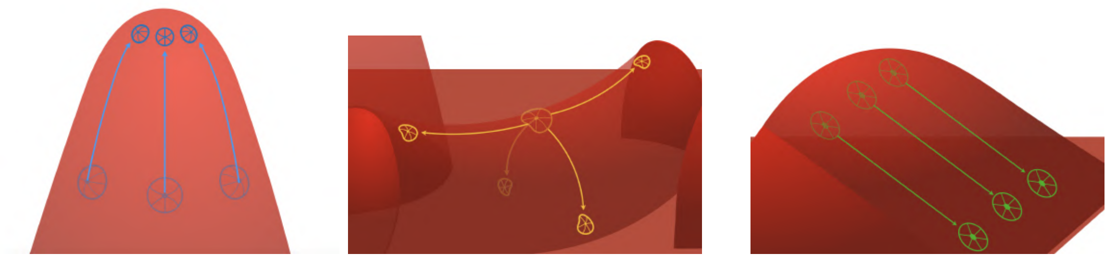


Riemannian Manifolds in 12 Minutes
PDF link: Riemannian Manifolds

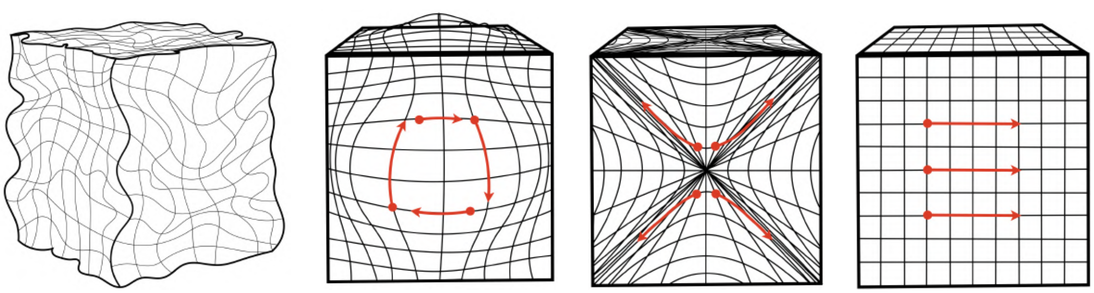


How to Describe an Entire Surface with Just Two Numbers
Gaussian Curvature

Ricci curvature is defined as the *contraction* of the *Riemann curvature tensor.* We will see very clearly what it means and what it looks like in a general space, but for now think of it as a way of measuring how areas, volumes, or hypervolumes (i.e., higher-dimensional volumes) expand or shrink as you move outward from a point.

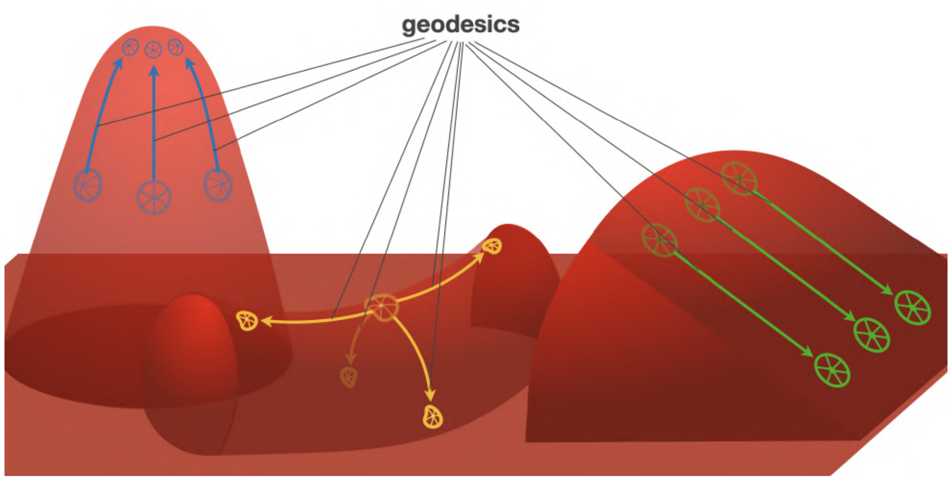


Ricci curvature at a point in space can behave in three distinct ways, depending on how *geodesics* respond nearby:
If it's positive, geodesics tend to converge. If it's negative, they diverge. If it's zero, they stay parallel, as if space were flat in that direction.

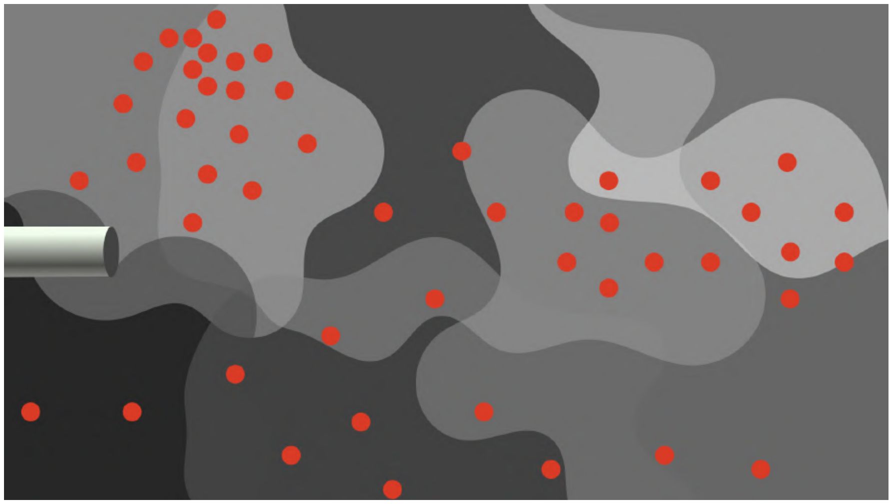


In some sense, a geodesic is the locally shortest path between two points – like a straight line, but on a curved space instead.

It's the path a particle follows if no forces act on it except the shape of the space itself. So, as you can imagine, it is very useful when trying to study the curvature of a space.



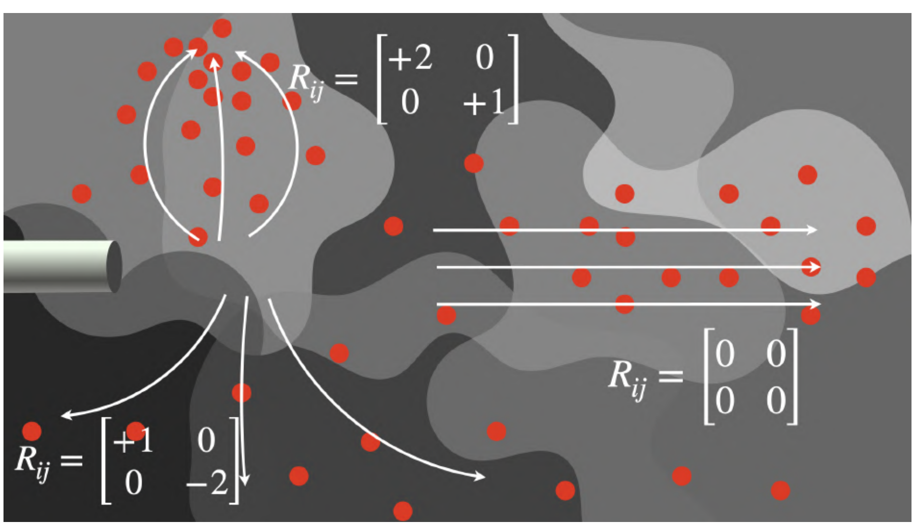
For example, imagine you're inside a foggy, curved universe, and you release a cloud of particles from a single point – all moving freely in different directions. As time passes, do they spread apart? Do they collapse together? Or do they just stay evenly spaced? This isn't a question of force.



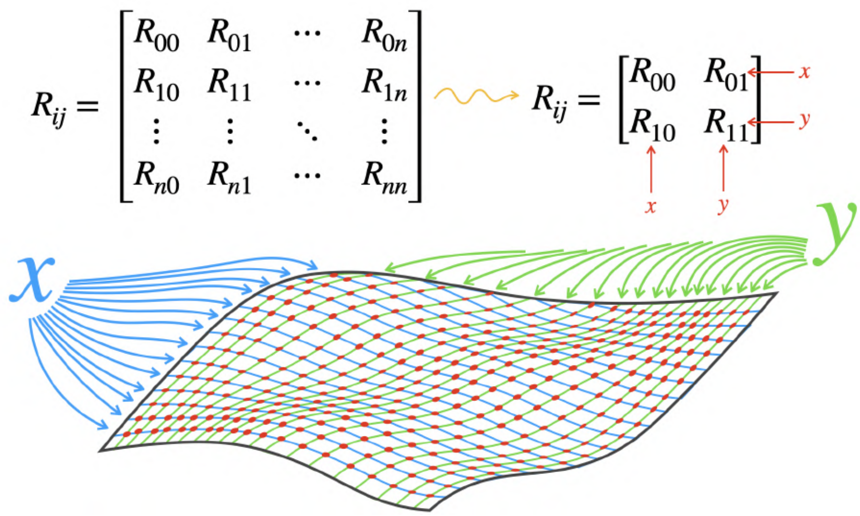
There is no force acting on them – it's a question of geometry, instead. And that's where the Ricci curvature tensor comes in.

$$R_{ij} = \begin{bmatrix} R_{00} & R_{01} & \cdots & R_{0n} \\ R_{10} & R_{11} & \cdots & R_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ R_{n0} & R_{n1} & \cdots & R_{nn} \end{bmatrix} \quad i,j \in \{1, \ldots, n\}$$

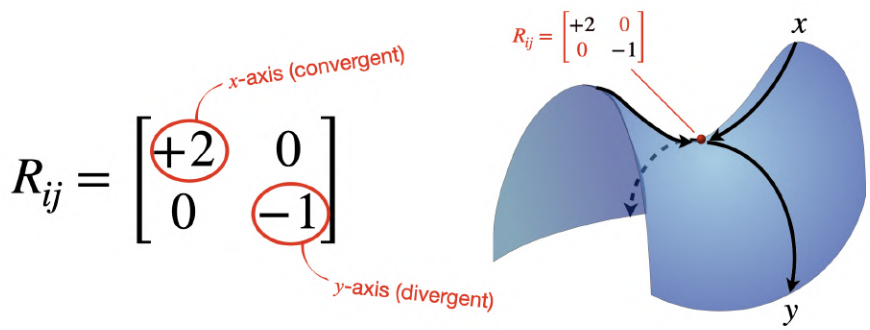
It tells you – at a specific point – how your cloud will behave. Whether it shrinks, stretches, or stays perfectly stable. It fixes the problem by translating the "invisible" bending of space into something measurable, i.e. the *change of volumes*.



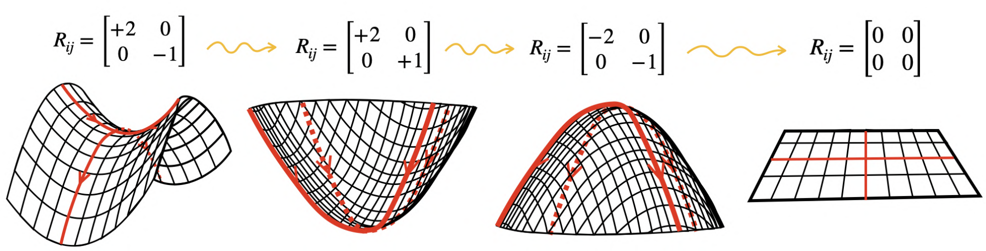
So far we've seen a representation of a general Ricci tensor for an n-dimensional space. In $2D$, this curvature tensor becomes a 2×2 matrix that describes how space curves at a single point.



For example, the following matrix (below) tells us that geodesics along the x-axis tend to converge (i.e., positive curvature), while those along the y-axis diverge (i.e., negative curvature). A point in a space that behaves this way is called a *saddle* point.



At this point, the curvature matches this matrix: compressing space in one direction and stretching it in the other. This is just one example, but you can easily imagine how this tensor would change, from point to point, for different types of curvature.



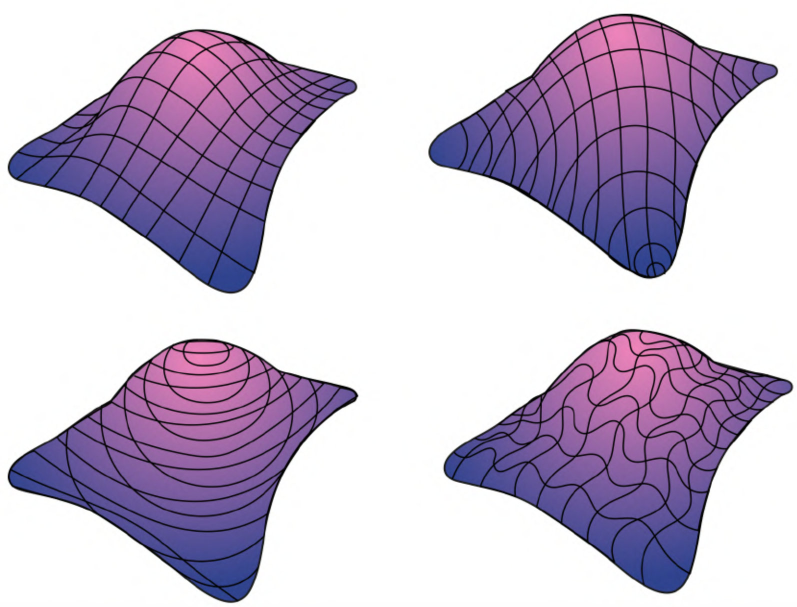
But… what about the off-diagonal terms R_{01} and R_{10}? They are all zero in these examples.

R_{01}

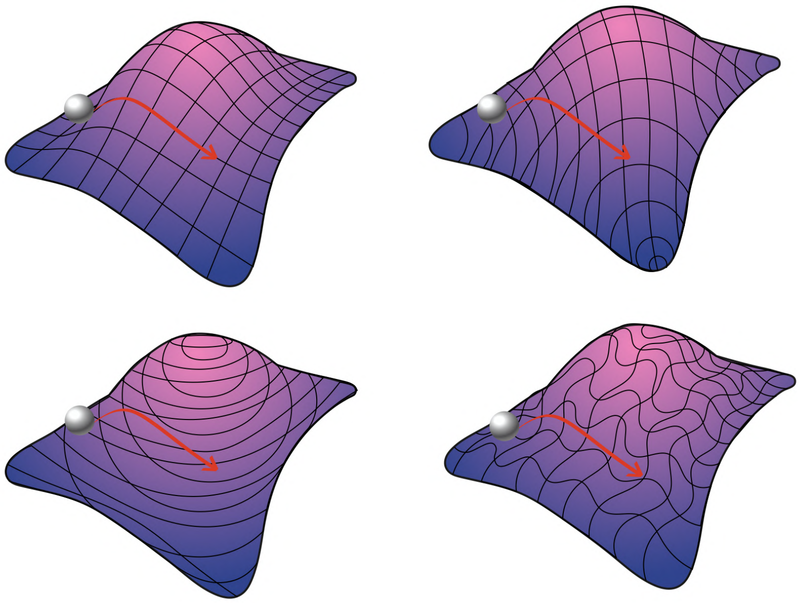
$$R_{ij} = \begin{bmatrix} +2 & 0 \\ 0 & -1 \end{bmatrix}$$

R_{10}

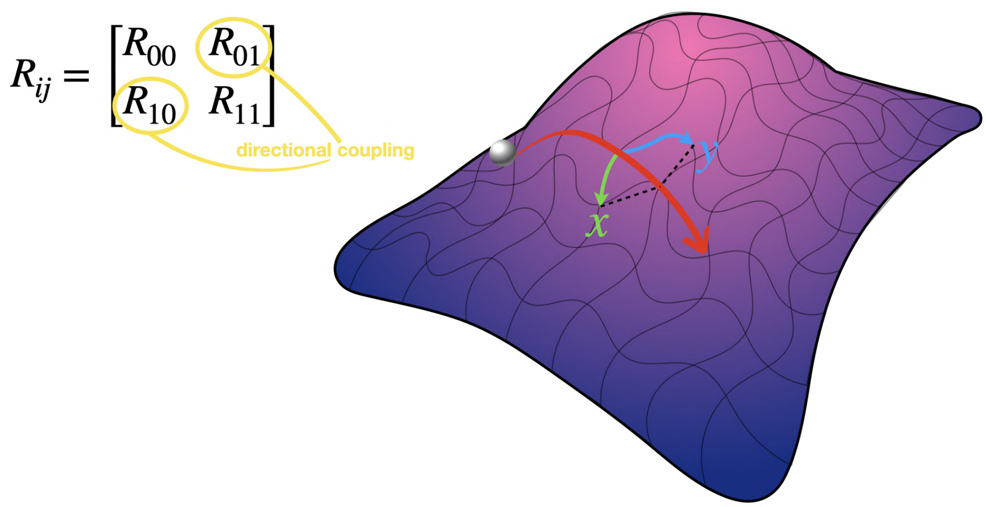
What would it mean if one of them were not zero? In that case, the chosen intrinsic coordinate system is not what we would naturally expect. This is the coordinate system defined on the surface itself, not from an embedding in $3D$ space. It is not what we would expect because the coordinate axes are not aligned with the directions of principal curvature.



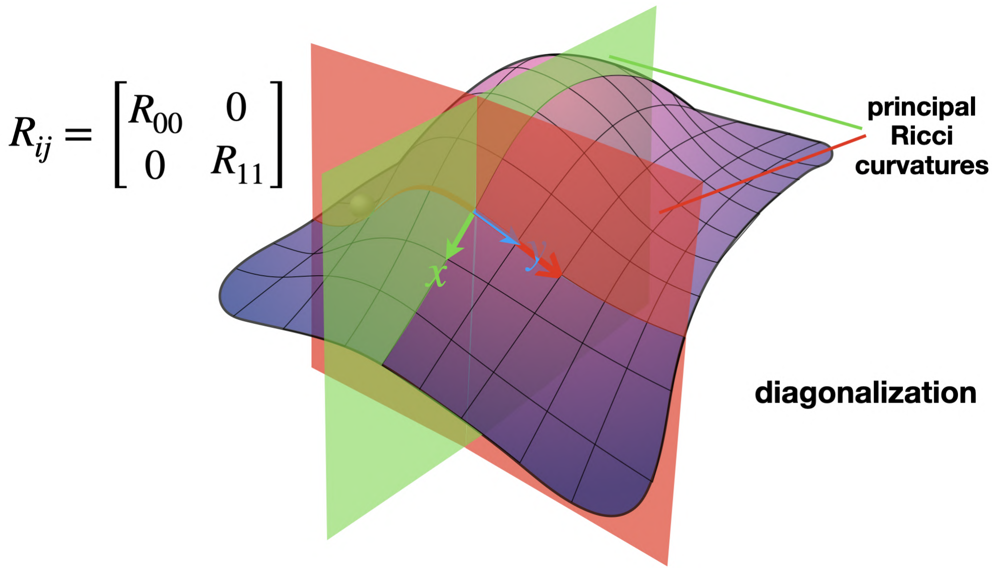
If you imagine releasing particles from a particular point, they would still follow geodesic paths, but these paths would not be aligned with your grid anymore.



Algebraically, the off-diagonal terms represent *directional coupling*: motion in one intrinsic direction (say x) contributes to the geodesic expansion or contraction in another (say y).



However, this is not an intrinsic property of the geometry itself, but rather of the coordinate frame you've chosen. The Ricci tensor is a geometric object – it exists independently of coordinates – but its matrix representation depends on the basis. If you change to a new intrinsic coordinate system aligned with the eigenvectors of the Ricci tensor at that point, the matrix becomes diagonal. This process is called diagonalization, and it reveals the *principal Ricci curvatures*, which are intrinsic, and therefore independent of the coordinate choice. Visually, this corresponds to rotating the perspective on the surface.



Let's see a concrete example: say that at a specific point on a curved surface, the Ricci curvature tensor is

$$R_{ij} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$$

This is a symmetric matrix, since $R_{ij} = R_{ji}$.

Remark: The Ricci tensor is always symmetric. This is a direct consequence of the way in which it is built, i.e. from the Riemann tensor via a contraction:

$$R_{ij} = R^{k}{}_{ikj} = R^{0}{}_{i0j} + R^{1}{}_{i1j} + \cdots + R^{n}{}_{inj}$$

We will not get into too many details here, but the Riemann tensor satisfies a special symmetry:

$$R^{k}{}_{ikj} = R^{k}{}_{jki}$$

This makes the Ricci tensor into a symmetric matrix.

Anyway, going back to our example, we can calculate the eigenvalues of this Ricci matrix transformation:

$$R_{ij} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \implies \det \begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = \lambda^2 - 2\lambda - 3 = 0 \implies$$

$$\implies \boxed{\lambda_1 = 3} \qquad \boxed{\lambda_2 = -1}$$

If you want to learn how to calculate the *eigenvalues* and *eigenvectors* of a matrix, as well as a very visual and intuitive interpretation of these concepts, check out the video and PDF below:

The Core of Eigenvalues & Eigenvectors
PDF link: Eigenvalues & Eigenvectors

Therefore, the diagonalized Ricci tensor is:

$$R_{ij} = \begin{bmatrix} +3 & 0 \\ 0 & -1 \end{bmatrix}$$

Which reveals itself as a saddle point!

Notice an interesting fact: even though the original Ricci curvature tensor looked positive in all directions $\left(R_{ij} = \begin{bmatrix} +1 & 2 \\ 2 & +1 \end{bmatrix} \right)$, it is just the illusion of the "bad" coordinate choice. Once diagonalized, we see the true story: curvature pulls space together in one direction and pushes it apart in another. The eigenvectors tell us those directions – they are intrinsic, geometric axes drawn right inside the surface's tangent space.

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