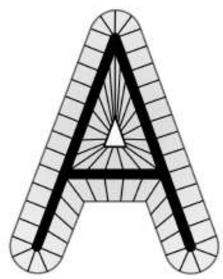
How to Get to a Homotopy Naturally

by DiBeos

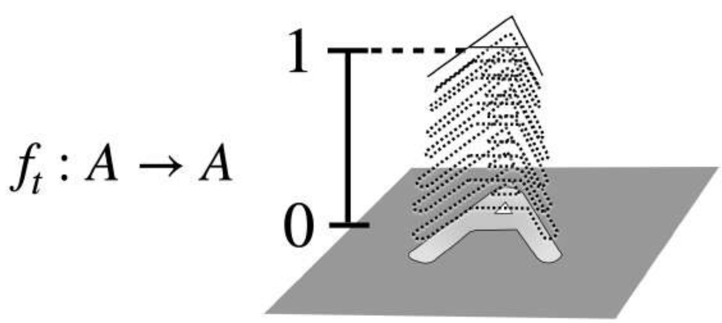
One of the key ideas of algebraic topology is to consider that two spaces are equivalent to one another if they have the same shape, but in a very interesting sense.

Let's take a very simple example, the letter A. We can create two different forms out of it: a thick form and a thin form. Topologically, they are one and the same even though they look different, and we can continuously move from one shape to the other. A nice way to do this is to create line segments that connect each point on the boundary of thick A to unique points on the thin subletter A.



We can continuously slide one shape into the other along the line segments. This process can happen during the time interval $0 \leq t \leq 1$. At $t = 0$ the A is in its 'thick' form, at $t = 1$ in its fully thin form.

The process is explained by the family of functions $f_t\colon A \to A$. The subscript t indicates that the function is not static but varies with time. Each f_t corresponds to a specific moment in the transformation process within the interval $[0, 1]$. Formally, $f_t(a)$ is the point to which a given point $a \in A$ has moved at time t.



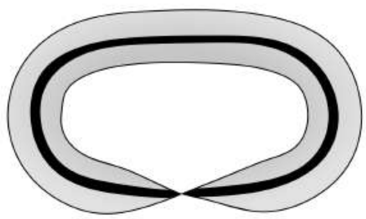
This continuity ensures that there are no sudden jumps or abrupt changes in the position of any point as time progresses from $t = 0$ to $t = 1$. Each point's movement is smooth, maintaining the geometrical integrity of the letter throughout the entire transformation.

Each point moves along a straight line at a constant speed towards its final destination. So, the constant speed and direct path ensure that the transformation is orderly and predictable.

This example leads us to a general definition: a **deformation retraction** of a general space X, unto a subspace A is a family of maps $f_t : X \to X,\ t \in I$ such that $f_0 = 1$, (the identity map), $f_1(X) = A$, and $f_t | A = 1$ for all t. The family f_t should be continuous in the sense that the associated map $X \times I \to X$, $(x, t) \longmapsto f_t(x)$ is continuous.

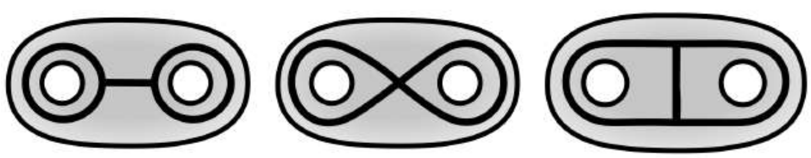
It's easy to find more examples similar to the one we started with, with the deformation retraction f_t being obtained by sliding along lines.

Here's an example of the deformation retraction of a Möbius strip, a one-sided surface, onto its core circle in a continuous way.



Speaking of continuous, to avoid overusing the word, we'll adopt the convention that maps between spaces are always assumed to be continuous unless we say otherwise.

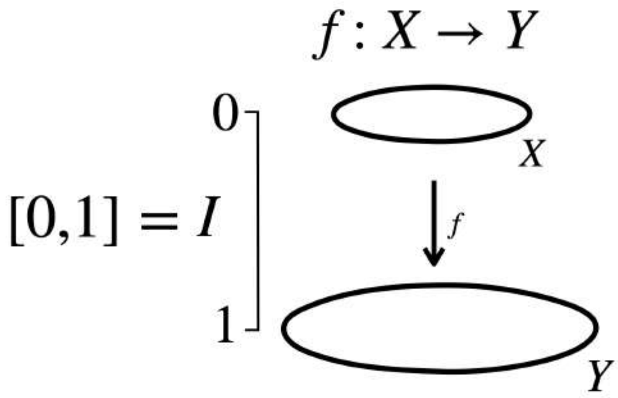
These figures illustrate another deformation retraction where a disk, with two smaller open subdisks, is continuously transformed into three separate subspaces.



Now that you have a good intuition of what this process looks like, let's finally describe it in a more general and abstract way.

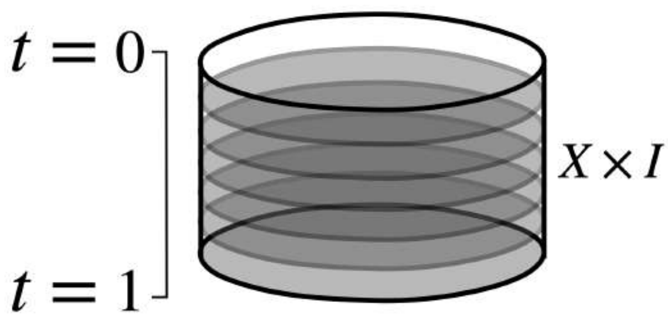
Let's say that we have a topological space X, which is much like the outer boundary of the thick letter A. Our goal is to transform X into another space Y, just as we wanted to transform the thick A into the thin A.

This transformation can be described by a family of functions $f : X \to Y$. Mathematically, this transformation is parametrized using the interval $[0, 1]$, denoted as $[0, 1] = I$.

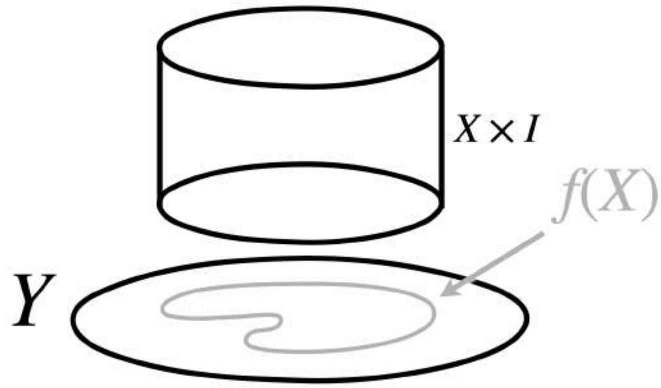


The interval I, and the space X, are "put together" by something called the Cartesian product $X \times I$. It consists of all ordered pairs $(x,\ t)$, where x is a single point in X, and t is a point in I. Each such pair represents a point x at a specific "time" t during the transformation.

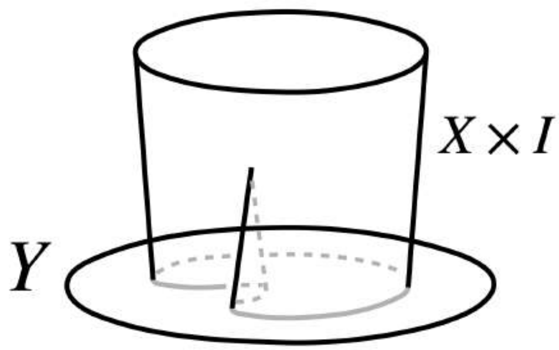
What happens is that you visualize each point in X "extending" from $t\ =\ 0$ to $t\ =\ 1$. For intuitive understanding, this extension can be thought of as tracing a line in a new dimension, analogous to the dimension of time. $X \times I$ then becomes a cylinder.



The cylinder can be visualized as containing all of the intermediate states of X from the beginning of the transformation at $t = 0$ to the end at $t = 1$, where X has been transformed into Y. Rigorously, $f_1(x)$ indicates the final position of each point x in X in the space Y at $t\ =\ 1$. That is why the cylinder $X \times I$ actually becomes $f(x)$ in Y;



f_1 essentially collapses or projects the entire transformation process down to the final configuration in Y.



In deformation retraction you often deal with a single function f expressed as a family f_t, where f_0 is the identity on X and f_1 is the retraction onto a subspace. The focus is often on how the space X can be continuously simplified or shrunk onto a subspace A without changing its essential topological features. The emphasis is on the transformation process itself rather than on showing equivalence between two completely distinct functions.

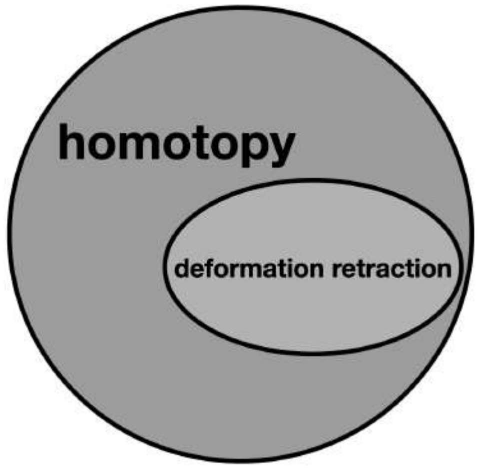
But, a deformation retraction is actually a special case of the general notion of *homotopy*, where now, you will need a second function g to be used as an endpoint.

When you have two continuous functions $f, g : X \to Y$, a homotopy between f and g is a family of continuous functions $H : X \times [0, 1] \to Y$ such that:

$H(x, 0) = f(x)$ for all $x \in X$ (so, at the beginning of the homotopy the function is f).
$H(x, 1) = g(x)$ for all $x \in X$ (so, at the end of the homotopy, the function is g).

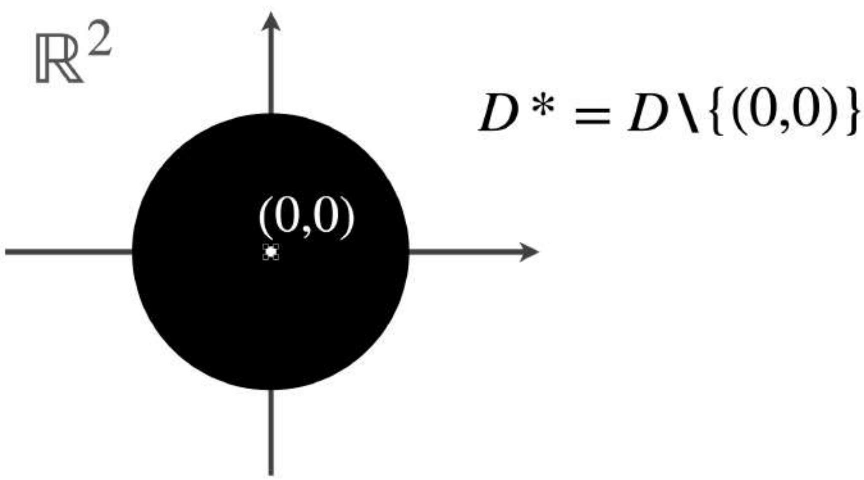
Essentially, a homotopy shows how one function can be continuously transformed into another, and it helps to define an equivalence relation on the set of all continuous functions from X to Y.

A deformation retraction and a homotopy look pretty similar. But, a deformation retraction is a specific type of homotopy that is used to simplify a space into a subspace while maintaining its topological essence. A homotopy – not necessarily.

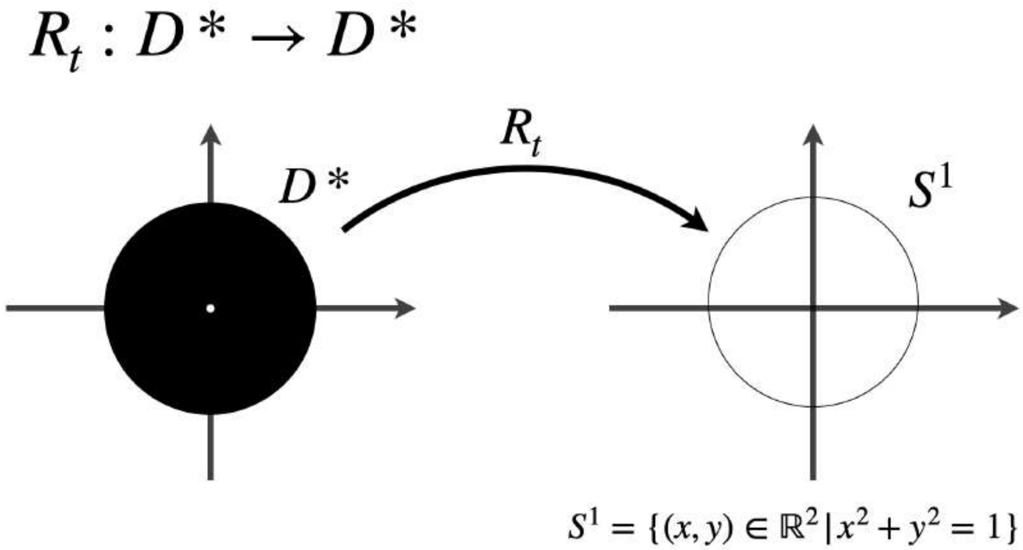


Now let's first see a concrete example of a deformation retraction.

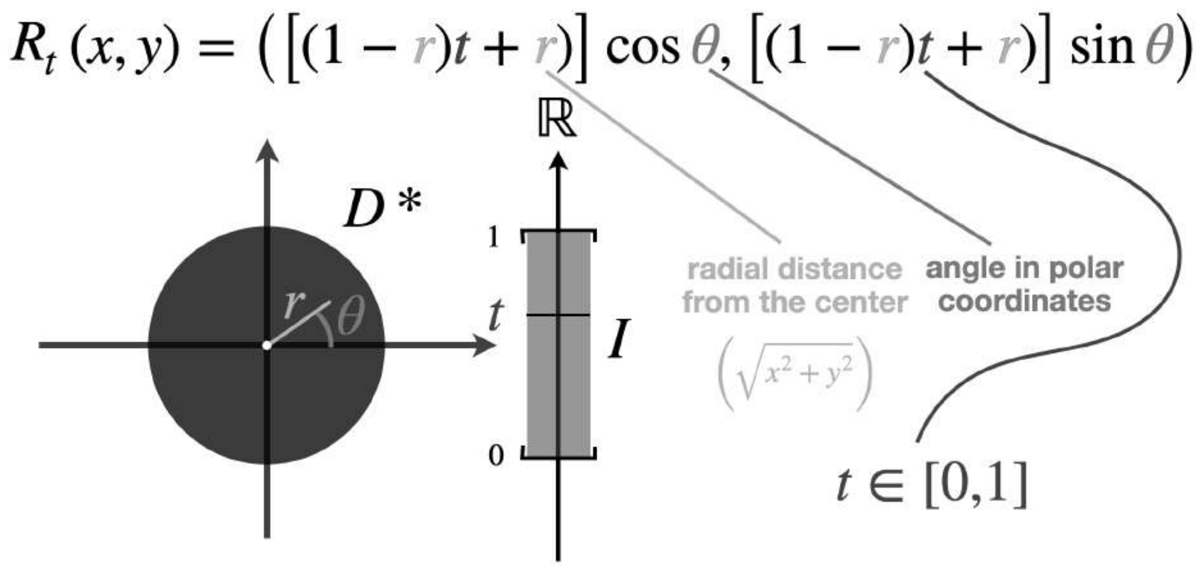
Consider the unit disk in $\mathbb{R}^2$, $D = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}$. Now, remove the origin $(0, 0)$ forming the punctured disk.



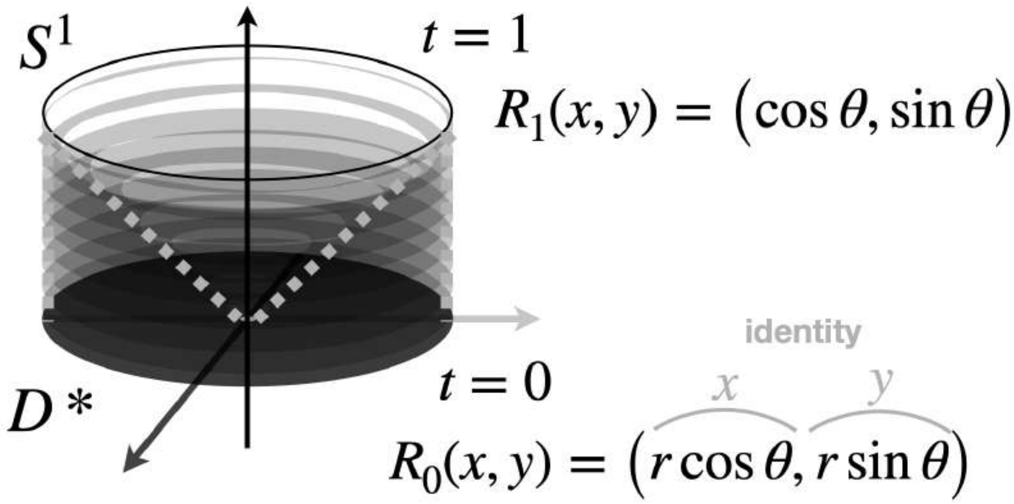
We define a deformation retraction $R_t: D^* \to D^*$ that continuously shrinks every point radically onto the boundary circle S^1.



We define R_t as the following:

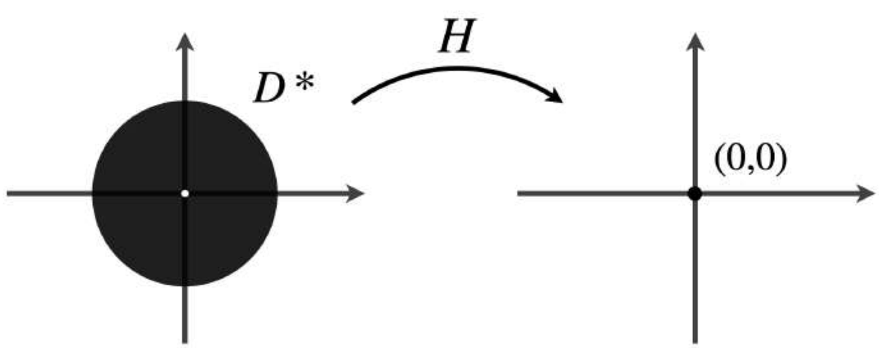


For $t = 0$, we get the identity, and for $t = 1$, we get the retraction onto S^1, which is exactly a projection onto the unit circle.



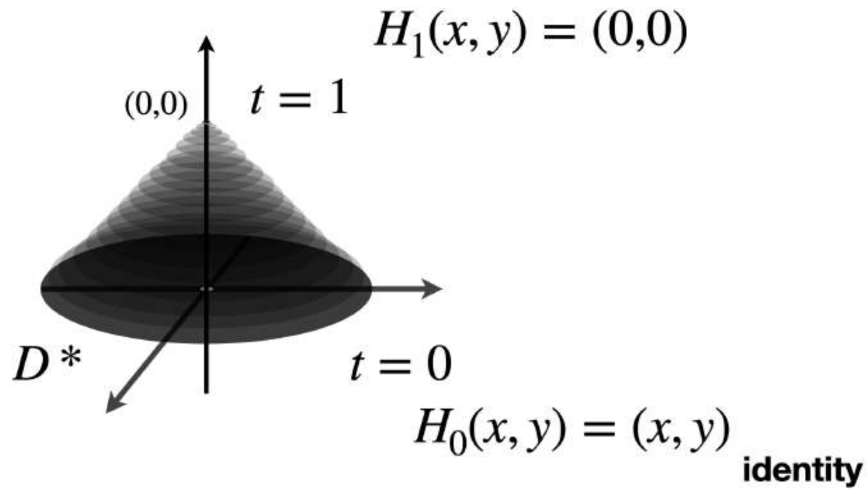
Now let’s see a concrete example of a homotopy that is not a retraction. Try to notice the difference between them.

Consider again the punctured disk. We now define a homotopy $H_t : D^* \to D^*$ that continuously shrinks every point toward the origin, rather than onto the boundary.



$$H_t(x, y) = \big((1-t)x, (1-t)y\big), t \in [0,1]$$

At $t = 0$ we have the identity, and at $t = 1$ every point collapses onto the origin.



Why is this not a deformation retraction? Well a deformation retraction requires that the space be continuously deformed onto a proper subspace like the unit circle, and that the subspace remains fixed throughout the homotopy. But the final space $\{(0, 0)\}$ is not a subspace of D^* because $(0, 0) \notin D^*$.

In contrast, in the deformation retraction we saw earlier, the final space which was the unit circle S^1, was a subspace of the initial shape, namely the punctured disk. Beyond that, a homotopy can collapse everything leaving no subspace fixed. But a deformation retraction cannot.

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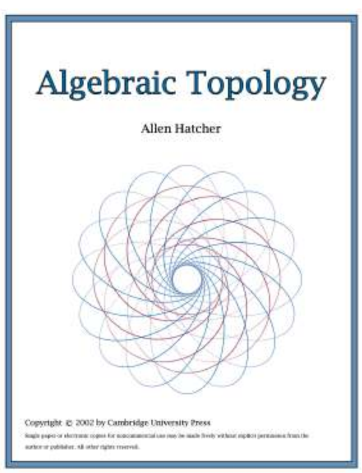
Inspired by the book Algebraic Topology



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