[Differential Geometry is Impossible Without These 7 Things]

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1) Parametric Representation of Curves and Surfaces

Normally, if we were to describe a circle of radius one, we do so like this $x^2 + y^2 = 1$. Here, the points (x, y) on the circle are defined indirectly: you must "solve" the equation or check a point against it. There's no explicit formula to produce points directly, and analyzing local properties (like slopes of tangents) requires solving equations each time.

But in the 1700's, Leonhard Euler and Gaspard Monge greatly advanced a different technique, called parametric representation.

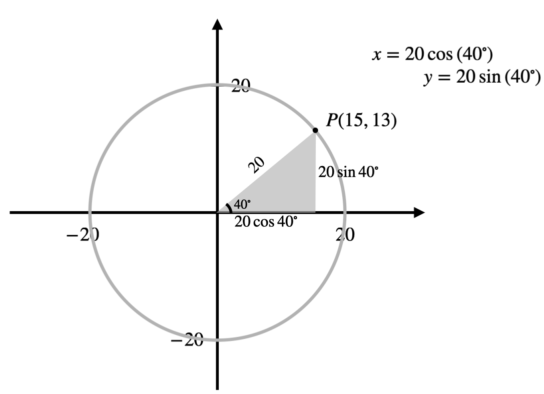
Parametric representation involves describing geometric shapes, like curves and surfaces, through smooth functions that depend on one or more parameters. Instead of describing a shape implicitly, we explicitly represent each coordinate as a function of one or more variables (these are known as parameters).

A general parametric form looks like this, which is a curve: $\mathbf{r}(t) = \left(x(t),\ y(t),\ z(t)\right)$, with one parameter (t).

For a surface, we have $\mathbf{S}(u, v) = \left(x(u, v),\ y(u, v),\ z(u, v)\right)$ this time with two parameters are $(u,\ v)$.

For the circle, we, instead, of representing it implicitly with $x^2 + y^2 = 1$, we can represent it explicitly: $\mathbf{r}(t) = (\cos t,\ \sin t),\quad 0 \le t < 2\pi$.

Let's take an example $(x,\ y) = (20 \cos t,\ 20 \sin t)$, we're explicitly describing every point on the circle as the tip of a vector of length (20) rotating around the origin. The parameter (t) represents the angle that the rotating vector makes with the x-axis. Take the coordinates of the point P: $x = 20 \cos\left(40^\circ\right),\ y = 20 \sin\left(40^\circ\right)$



Parametrically, you're explicitly generating points as you smoothly move the angle t around the circle.

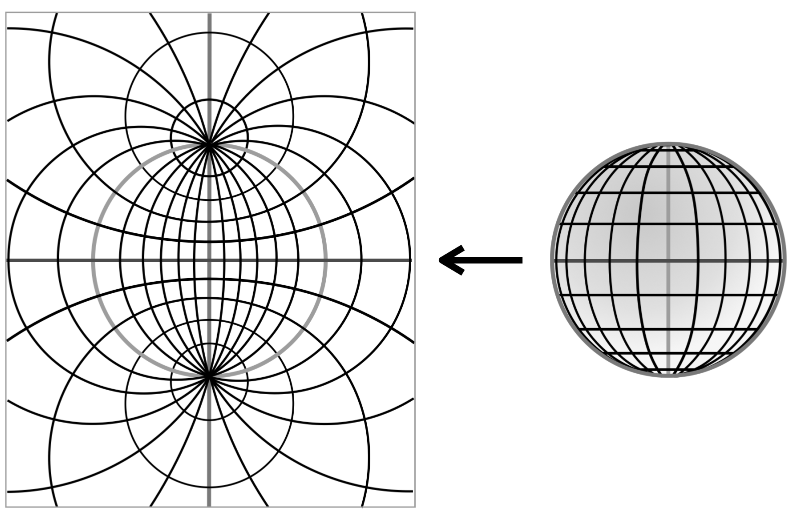
Before Euler and Monge, geometry was primarily studied using classical Euclidean tools. Parametric representation brought us the ability to visualize how shapes form and change continuously. Plus, it clarified definitions of tangent vectors and tangent planes.

2) Gauss's Theorema Egregium (or Remarkable Theorem)

When we think of a surface, we usually think of it as embedded in a space, but in 1827, in his paper "Disquisitiones generales circa superficies curvas", Gauss proved that Gaussian curvature, denoted as K, of a surface is an *intrinsic* property. Intrinsic means that curvature depends solely on distances measured on the surface itself—completely independently of how the surface is embedded in 3D space.

Simply put: even if you bend or reshape the surface (without stretching it), the Gaussian curvature at each point remains the same. Surfaces could be studied completely internally, without reference to a surrounding space.

Consider the sphere, which is naturally curved everywhere. Gaussian curvature K is positive everywhere, specifically $K = 1/R^2$ for a sphere of radius R. Because it's positive, the sphere *cannot* be flattened into a plane without stretching. An example is that map of the Earth, which is distorted when flattened.



If we take a cylinder, it's visually curved when we see it in 3D space, but it's not "truly curved" because you can flatten it out into a rectangle without distorting it, known as an isometric transformation. Its Gaussian curvature is $K = 0$.

Thus, surfaces could now be classified by intrinsic curvature.

$K > 0$ (are sphere-like surfaces),
$K = 0$ (are flat surfaces like planes and cylinders),
$K < 0$ (are saddle-shaped surfaces).

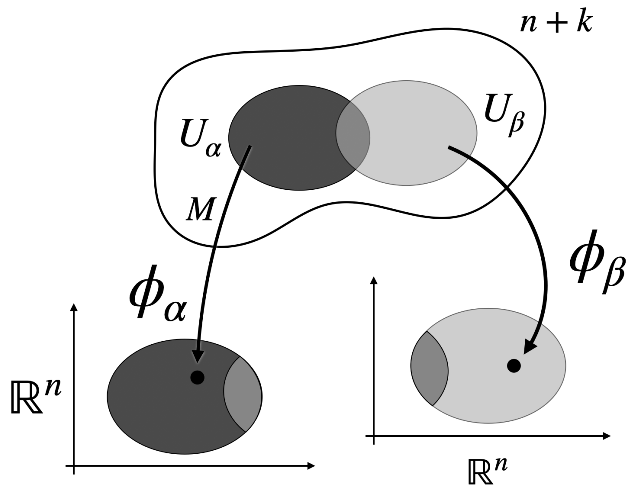
Thus, Gauss allowed for surfaces to be studied independently of their embedding and become a modern abstract discipline.

3) Riemann's Generalization of Geometry

Created by Bernhard Riemann, Riemannian geometry studies abstract spaces, known as manifolds, defined locally by a smoothly varying object called a metric tensor. Riemann took Gauss's idea (that geometry could be studied intrinsically) and generalized it.

Gauss's Theorema Egregium was revolutionary but still limited to 2-dimensional surfaces embedded in 3-dimensional space. Riemann filled this gap and generalized it so radically that geometry became about abstract shapes, and spaces, in any dimension.

A sphere, considered as a Riemannian manifold, is locally very similar to a flat plane. At every point on the sphere, you can create a small, flat "map" or coordinate chart — just as small regions of the Earth's surface appear flat to us locally. By combining many of these small charts, you form what's called an atlas, which smoothly covers the whole surface.



Each chart comes with what's known as a metric tensor, a smoothly varying inner product that tells you how to measure distances and angles directly on the sphere, independently of any external coordinates or embeddings.

For example, in spherical coordinates (θ, φ) the metric tensor at a point on a sphere with radius R is given by this:

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{pmatrix}$$

With this, distances, angles, and curvature are computed intrinsically, without the need to embed the surface. This can be done in limitless dimensions.

4) The Christoffel Symbols

Riemann provided the broad concept of manifolds and intrinsic metrics, but ~~it~~ didn't yet offer a straightforward way to explicitly differentiate vector fields on curved spaces.

This was solved by Elwin Bruno Christoffel, who developed Christoffel symbols, (often denoted as $\Gamma^{k}_{\ ij}$. They are mathematical objects that encode how coordinate systems change as you move around on curved spaces (or manifolds).

More intuitively speaking, they provide the machinery for comparing vectors located at different points of a manifold, defining what it means for vectors to be "parallel," and describing how vectors change directions intrinsically.

Formally, they are defined in terms of the metric tensor g_{ij} of the manifold:

$$\Gamma^k_{ij} = \frac{1}{2} g^{k\ell} \left(\frac{\partial g_{\ell i}}{\partial x^j} + \frac{\partial g_{\ell j}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right)$$

These symbols serve as coefficients in the definition of covariant derivatives, a generalization of ordinary derivatives to curved spaces.

Consider again a sphere of radius R in spherical coordinates $(\theta,\ \varphi)$

The metric tensor is this

$$g_{ij} = \begin{pmatrix} R^2 & 0 \\ 0 & R^2 \sin^2(\theta) \end{pmatrix}$$

Using Christoffel's formula, we can compute the symbols. For example, a few Christoffel symbols for this sphere are:

$$\Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta, \quad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta.$$

They tell you explicitly how vectors change when you move along certain directions, enabling you to determine the shortest paths (known as geodesics) or the intrinsic acceleration (or curvature effects).

For example, when navigating on Earth, Christoffel symbols would explicitly tell you how a direction (like "north") changes as you move along a parallel (latitude line) — even if you feel you're moving straight, your compass direction changes intrinsically due to curvature.

5) Tensor Calculus (Absolute Differential Calculus)

Tensor Calculus organizes information about vectors, matrices, and more general multi-dimensional entities into objects called tensors, whose properties remain consistent and invariant under coordinate transformations.

Tensors generalize scalars (which are 0-dimensional), vectors (which are 1-dimensional), and matrices (which are 2-dimensional) to higher dimensions.

Christoffel symbols made differentiation of vector fields and curvature calculations practical but were not fully coordinate-independent themselves. They depend explicitly on the choice of coordinate system, but tensors are explicitly coordinate-independent.

Consider again the sphere or any curved manifold. The Riemann curvature tensor, denoted by $R^{i}_{\ jkl}$, holds all the information about how space curves at every point.

Formally, the curvature tensor is given in terms of Christoffel symbols as:

$$R^i_{jkl} = \frac{\partial \Gamma^i_{jl}}{\partial x^k} - \frac{\partial \Gamma^i_{jk}}{\partial x^l} + \Gamma^i_{mk}\Gamma^m_{jl} - \Gamma^i_{ml}\Gamma^m_{jk}.$$

If you pick a different coordinate system, the tensor's numerical components might change, but the tensor itself describes the same intrinsic curvature.

From this curvature tensor, you can derive simpler tensors: Ricci curvature tensor: $R_{ij} = R^{k}_{\ ikj}$ important in general relativity. And Scalar curvature: $R = g^{ij}R_{ij}$ giving an intrinsic measure of curvature magnitude. For example, the curvature tensor on a sphere of radius R simplifies considerably—reflecting its constant curvature everywhere.

6) Covariant Differentiation

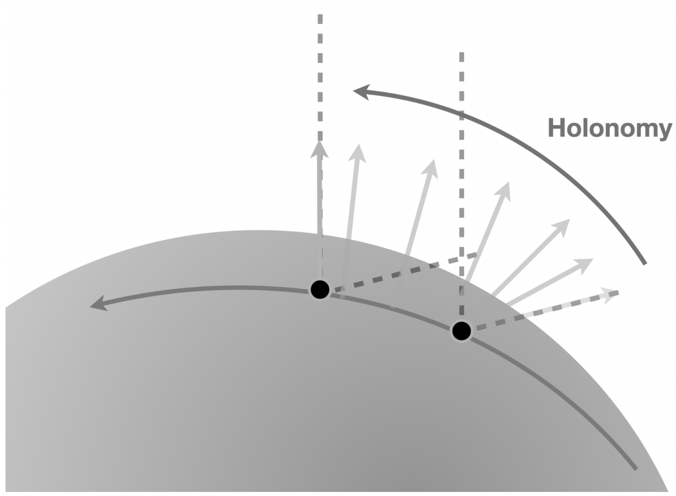
In flat spaces, ordinary derivatives work fine. But on curved spaces, straightforward derivatives aren't enough: vectors at different points live in different tangent spaces, so you can't simply subtract them or take differences in the usual sense. You need a way to "transport" vectors smoothly and intrinsically. Covariant differentiation solves this problem.

Developed by Gregorio Ricci-Curbastro and Tullio Levi-Civita, covariant differentiation is a method of differentiating vectors and tensors along curved manifolds in a manner that respects the manifold's intrinsic geometry.

Formally, covariant differentiation of a vector field V^i along the coordinate direction x^j is expressed as:

$$\nabla_j V^i = \frac{\partial V^i}{\partial x^j} + \Gamma^i_{jk} V^k$$

Covariant differentiation tells you exactly how vectors must change as you move along a path to remain intrinsically parallel. On the sphere, it predicts how much your arrow turns relative to your original direction after traveling along certain paths—this is known as holònomy.



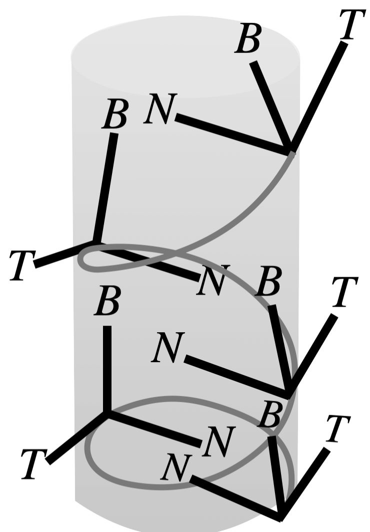
Mathematically, covariant differentiation describes this as the condition: $\nabla_{path} V^i = 0$ indicating the vector V^i is parallel transported and not changing intrinsically along the path.

7) **Cartan's Method of Moving Frames**

Covariant differentiation still relied heavily on coordinate systems and algebraic complexity. This messed with its intuitive geometric meaning. So, in the early 1920s, Élie Cartan introduced his method of Moving Frames.

It's a geometric approach that studies curves, surfaces, and higher-dimensional manifolds through carefully chosen local coordinate frames.

Consider a curve in 3D space—for example, a helix winding around a cylinder. At each point on this helix, Cartan's moving frame gives us a set of three orthonormal vectors (the "Frenet-Serret frame")



The **tangent vector** (T) points along the curve's direction.
The **normal vector** (N) points inward toward the curve's "center of curvature."
The **binormal vector** (B) is perpendicular to both (T) and (N), giving a third spatial direction.

Formally, a *frame* at each point on a manifold is just a collection of vectors forming a basis for the tangent space. As you move on the manifold, this basis smoothly changes—this motion is described by differential equations known as Cartan's structure equations.

Cartan's equations explicitly capture this motion

$$\frac{d}{ds}\begin{pmatrix} T \\ N \\ B \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix} T \\ N \\ B \end{pmatrix}$$

Here, κ is curvature, τ is torsion, and s is the arc-length parameter. Notice how simple and elegant these equations are—revealing precisely how the curve's geometry evolves intrinsically.

Cartan's method of moving frames was the final innovation that made differential geometry fully intrinsic, visual, and intuitive. And these were the discoveries that allowed us to get to Differential Geometry.

If you'd like to know more about differential geometry, check out this video:
The Core of Differential Geometry

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