Classification Theorem

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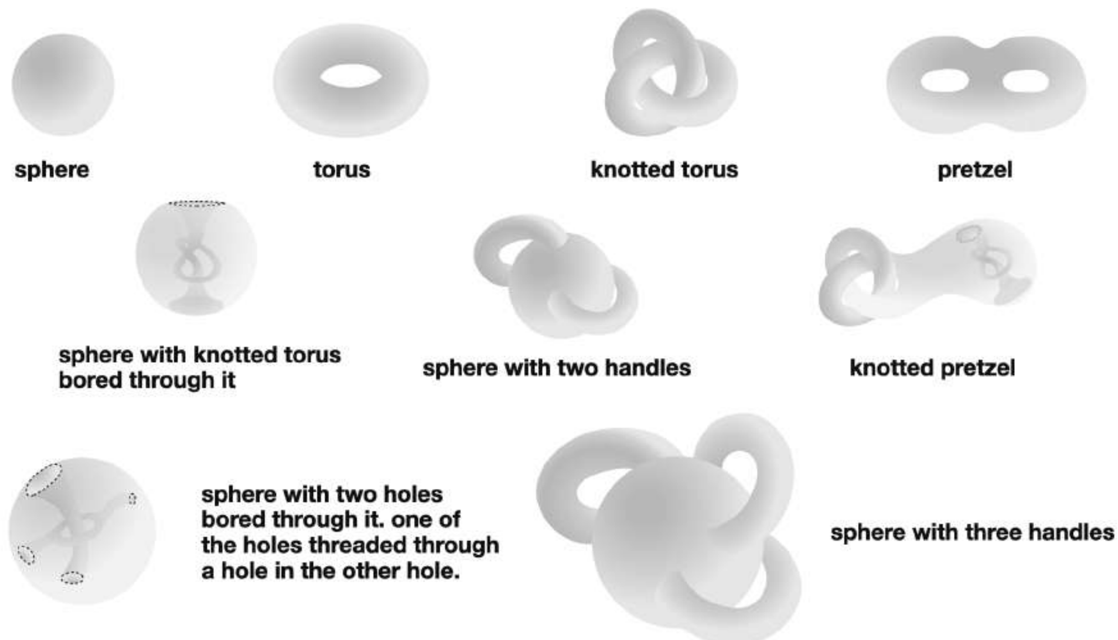
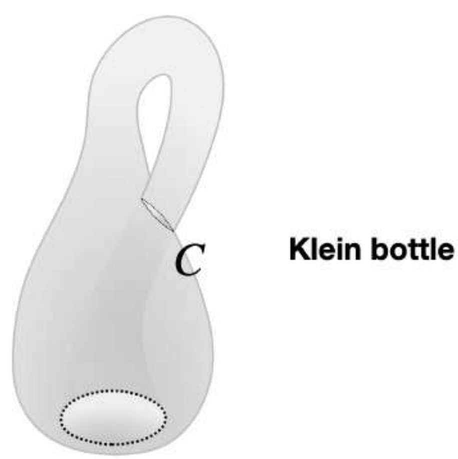
This video was based on this article by E. C. Zeeman

Let's look at a few examples of surfaces. All of these are surfaces in 3 dimensions, and none of them are solid inside.



This is a surface which looks similar, but is quite different from the properties of the other shapes. It is a Klein bottle. It's different from the others because it's the only surface that appears to intersect itself, here in the circle C.

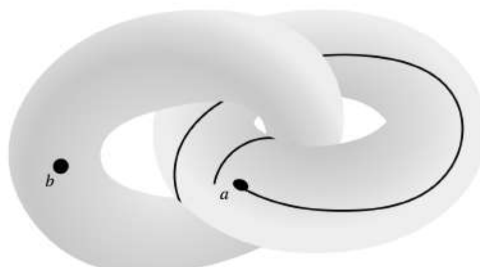


The Klein bottle is formed by taking a cylinder, narrowing one end like a bottle, bending it to the side, poking it through, then widening that end, and sewing it onto the bottom from the inside. We'll talk more about the Klein bottle a little later.

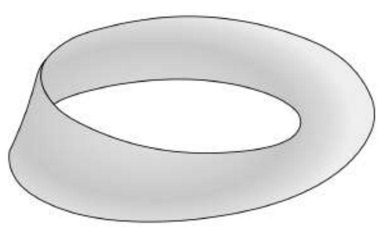
All of the 10 surfaces we just drew have three properties in common.

First, they are **connected**. They are all in one piece. A definition which is equivalent says that any two points on the surface can be joined by one path on that surface.

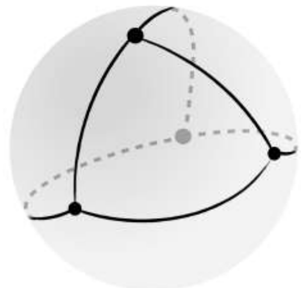
These two tori for example are *not* connected.



The second property is that they are **closed**. Our usage of the word closed is different from that of “open and closed sets” which are usually the definitions you encounter at the beginning of analytic topology books. Instead, we mean that there’s no boundary or rim. A very clear example of a surface that is *not* closed is a Möbius strip, which is formed by taking the ends of a strip and sewing them together “the wrong way”. If there was a bug crawling on one side, and it went all the way around, it would end up on the other side.

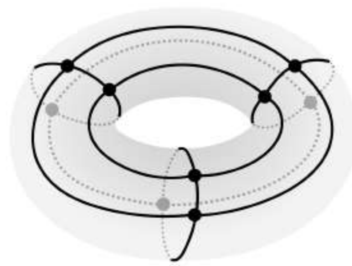


And the third property is that they are **triangulable**, which means that we can cut up the surface into a finite number of vertices, edges, and faces. If the surface is curved, the edges would of course also have to be curved, but it’s possible to make a model in which they appear to be straight. For example this sphere is triangulated with 4 vertices, 6 curved edges, and 4 curved triangles, so that its corresponding model that is straight is actually a tetrahedron.

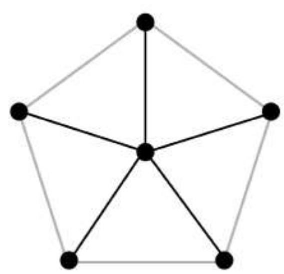


There are lots and lots of ways to chop up a sphere, and if we wanted to we could even use a million triangles to do so. The main thing to stress is that it can be done using a finite number of vertices, edges and faces.

Triangles aren't the only thing you can use to chop up a surface. For example, in order to chop up a torus, you'd need 9 vertices, 18 edges, and 9 squares. in which case we're essentially using polygons rather than triangles.

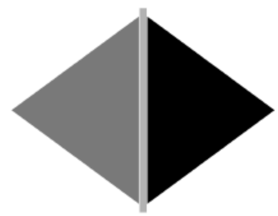


If we choose to chop up a surface using polygons, we can simplify into triangles by placing a vertex in the center of each polygon.

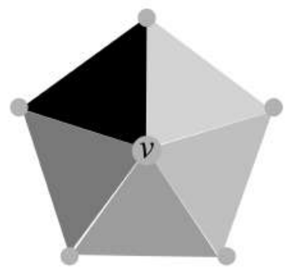


There's something special about triangles. Chopping up surfaces into triangles is so special in fact that it has a name: **triangulation**.

Any triangulation of a surface has two properties. The first is that, any edge you will find will be an edge of exactly 2 triangles.



And 2, any vertex, which we label v, is the vertex belonging to at least three triangles, and all the triangles that share the v fit round into a cycle.

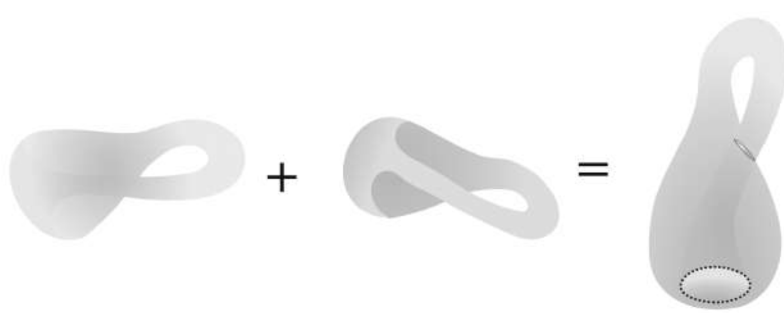


Your intuition will tell you correctly that indeed, any surface can be triangulated. But to actually prove this we'd need a lot of analytic topology, which is outside of our scope, so we'll have to assume that "triangulability" is true. Triangulation gives us a huge advantage because it helps us to reduce our initial task of classification to a combinatorial problem that is finite. Which can then be dealt with using finite mathematics.

So from now on, we will assume that all of our surfaces fulfill the three properties we spoke of (connected, closed triangulable).

Now, 9 out of the 10 shapes share another property called **orientability**. A surface is orientable if it does not contain a Möbius strip. If it does, it is **non-orientable**. The Klein bottle is of course the only one which is non-orientable out of these 10.

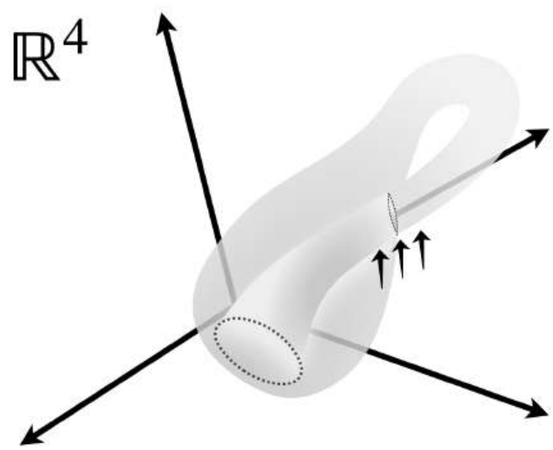
Imagine we could slice the Klein bottle in half. If you pay attention, the self-intersection circle C become two semicircles. If we lift it up a little from the inside and push down a little from the outside, we can clearly see that each piece is a Möbius strip. In fact the Klein bottle is the union of two Möbius strips sewn together along their boundaries.



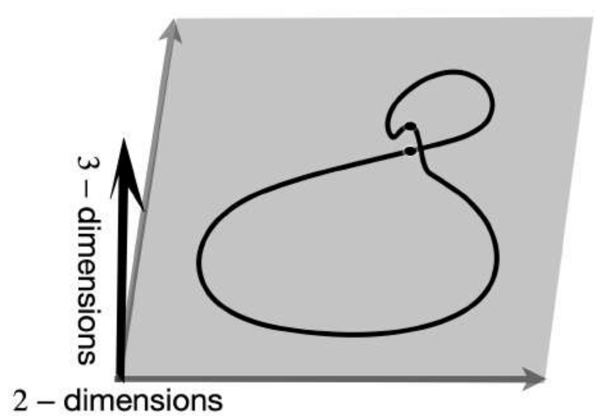
As you may well know, a Möbius strip is one sided. Some even call the Klein-bottle one-sided and claim that it does not have an inside, but this isn't true because of the "self-intersection" circle C. If we were to put a bug on it, it wouldn't be able to crawl from one side to the other because it would get stuck at C. It also wouldn't be able to crawl from the inside to the outside.

So, the Klein bottle cannot be constructed in 3 dimensions without self-intersecting. This difficulty arises for any surface which is closed and non-orientable. This difficulty doesn't arise for the Möbius strip only because it is *not closed*.

But in 4 dimensions, it is entirely possible to construct a Klein bottle without self-intersection. This is done by lifting the thin tube a little bit into the 4th dimension, and the self-intersection will disappear. That's where the concept of the Klein bottle having an inside becomes entirely meaningless.

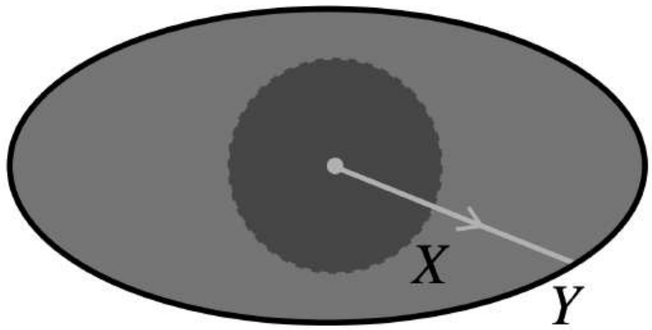


I know it's kind of hard to imagine, so let's try to understand it with this analogy: take a curve in 2 dimensions with a single self-intersection point. We can get rid of this point by lifting one of the branches a little into the 3rd dimension.

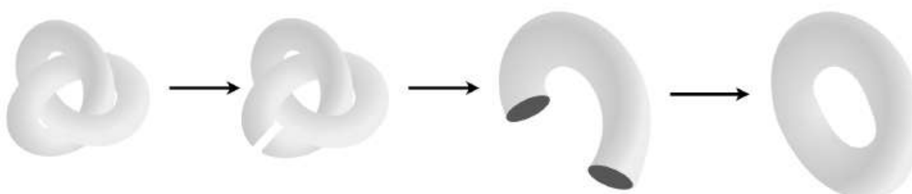


It therefore becomes meaningless to ask what becomes inside of the figure, because curves in 3 dimensions have no such thing as insides or outsides. The exact same thing happens when we lift the Klein bottle into the 4th dimension, and it becomes meaningless to talk about it being one-sided in either 3 or 4 dimensions. That's why the term "non-orientable" is preferred to the term "one-sided".

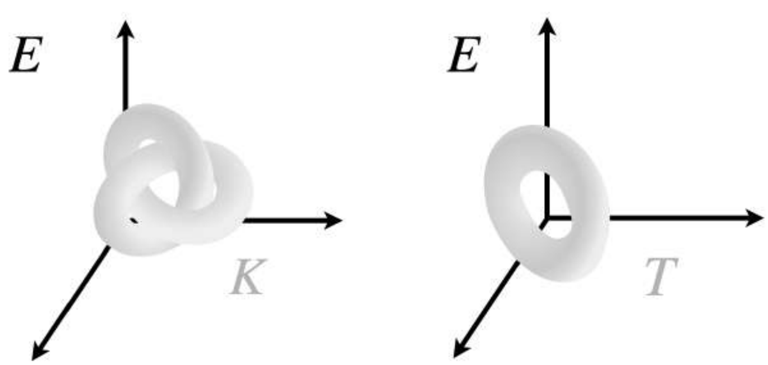
This brings us to the concept that distinguishes topology from any other type of geometry: **homeomorphism**. Simply, imagine a sphere X to be a balloon, and we can expand it into ellipse Y without cutting or gluing. $X \cup Y$ because each point on X moved to a unique point on Y. This action gives us a function $X \to Y$. It is continuous, which means that we did not cut anything, and it is one-to-one, so we did not glue anything.



Now, let's suppose that we indeed made a cut of a surface X, and later sewed it up exactly as it was before, but the result would be a surface homeomorphic to X.

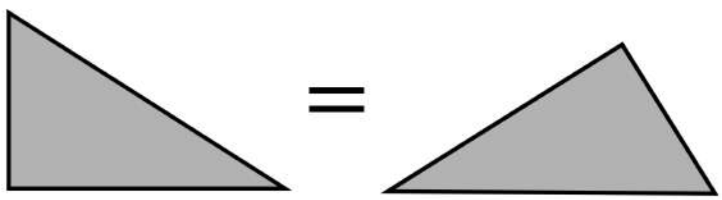


But this is actually a form of "cheating". Let's label the knotted torus K, the unknotted torus T, and the 3-dimensional Euclidean space in which they are embedded E. Now, we know two things to be true: The first that we know that there exists a homeomorphism between K and T ($K \cup T$). The second, that the way K is knotted is not actually a property of K itself, but rather it is a property of the way it is embedded in E.



So far we've only talked about the surface by themselves, and did not mention the harder problem of how many ways they can be knotted in E, which is still an unsolved problem by the way. Thus, to embed them in E is sometimes pretty confusing, and can sometimes raise weird things like the knotting of a torus, and the self-intersection and the one-sidedness of a Klein bottle. Because of that, we have to think of a surface as something abstract which exists on its own, *without* being embedded in anything. How this works is pretty hard to grasp without deeper knowledge of analytic topology. Because of that, we'll continue to base our intuition as though the surfaces are in E.

Finally, let's get to what classification means. Surfaces are classified by the invention of a list of standard surfaces. By proving that every surface is actually just homeomorphic to one of the standard surfaces, we classify them. A more rigorous definition says that a homeomorphism is an equivalence relation on the set of all surfaces, and the equivalence classes are listed. This list is actually surprisingly easy to compile. But the same can't be said about Euclidean geometry for example, because there two surfaces are equivalent if and only if a surface can be moved into the other by rigid motion. Therefore this list would be enormous for Euclidean geometry.

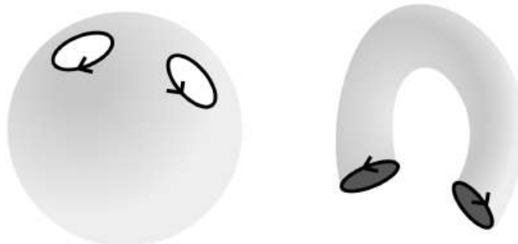


Oddly enough topology deals with situations that are much more complicated than geometry, but it does so with greater simplicity, because it mostly focuses on the more dramatic properties of a surface.

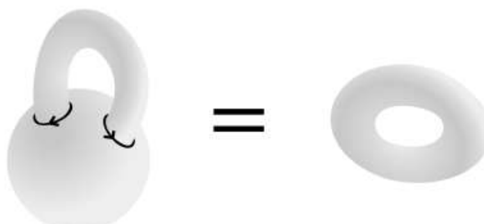
For example, even though a torus has an inner diameter and outer diameter, and other complicated things about it, the thing topology cares about is the fact that a torus has a hole, and that it persists in having one however we go about bending it. The number of holes a surface has is informally known as a **genus**.

In 4 dimensions, the difference between topology and geometry becomes even more pronounced, since geometry becomes almost completely algebraic, while topology actually becomes more geometric in a way.

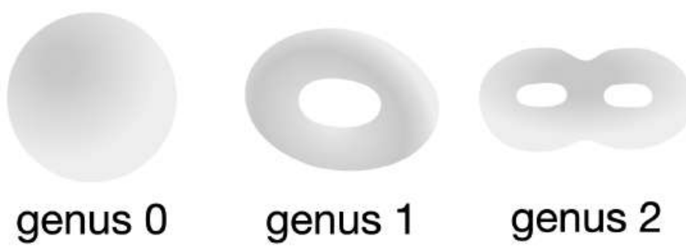
Now, to sew on handles to a sphere, we punch two holes in the sphere, take a cylinder, and sew its ends to the boundaries of the holes.



If we were to take a closer look, we'd actually find that there are arrows on the boundaries that show which way things should be sewn together. The arrows on the cylinder go the same way, but the arrows on the boundaries of the holes actually go opposite ways. This sphere with a handle is homeomorphic to a torus. If one of the arrows was reversed, it'd actually be homeomorphic to a Klein bottle.



In the beginning, we actually saw spheres with 2 or 3 sewn on handles. With this, we define **the standard orientable surface of genus n** $n \geq 0$ to be a sphere with n sewn on handles. This translates that genus 0 means sphere, genus 1 means torus, and genus 2 means pretzel.



The standard non-orientable surface of genus n ($n \geq 1$**)** is actually defined to be a sphere with a Möbius strip sewn on. This is a little hard to visualize in 3 dimensions because the resulting surface intersects itself, and it's a whole explanation in itself since there are various interesting ways to do this.

Thus we're finally able to state the classification theorem: any connected closed triangulable surface is homeomorphic to one of the standard ones.

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