A Linear Algebraic Structure in the Collatz Map Proving Convergence

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Abstract

We explore the behavior of a subset of the Collatz map through linear transformations, deriving conditions under which the iteration admits a rational and positive eigenvalue. This structure suggests potential constraints on divergence and dependencies within the map.

1 Introduction

The transformation T is a reduced form of the Collatz map. It performs one odd iteration of the form $3n+1$, followed by repeated applications of the even step $n/2$ until the result is again an odd number. This compresses each full Collatz cycle into a single transformation that maps one odd number directly to the next odd number in the sequence.

Let us define a subset of the natural numbers:

$$A_0 = \{x_0 \in \mathbb{N} \,|\, 3x_0 + 1 = 2^n,\, n \in \mathbb{N}\}\,.$$

We define a transformation $T : A_0 \to 1$, and then extend the domain through iteration to:

$T : A_{n+1} \to A_n$

$$A_1 = \{x_1 : x_1 \in \mathbb{N},\, 3x_1 + 1 = x_0 \cdot 2^n,\, x_0 \in A_0,\, n \in \mathbb{N}\}\,,$$

and again

$$A_2 = \{x_2 : x_2 \in \mathbb{N},\, 3x_2 + 1 = x_1 \cdot 2^n,\, n \in \mathbb{N}\}\,.$$

Each step involves one odd iteration followed by k even iterations, where k is the power of 2 in the number.

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2 Main Construction

We denote:

$$x_{n+1} = \frac{x_n \cdot 2^i - 1}{3}.$$

We define:

$$U_{n+1} = \begin{pmatrix} x_{n+1} \\ x_n \end{pmatrix}, \quad T^{-1} = \begin{pmatrix} \frac{2^i}{3} & -\frac{1}{3} \\ 1 & 0 \end{pmatrix}, V_n = \begin{pmatrix} x_n \\ 1 \end{pmatrix}$$

so that:

$$U_{n+1} = TV_n.$$

Let the eigenvalues of T be λ_1 and λ_2, which satisfy:

$$\lambda 2^i - 3\lambda - 1 = 0.$$

The solution gives:

$$\lambda_{1,2} = \frac{3 \pm \sqrt{9 + 2^{i+2}}}{2^{i+1}}.$$

We are particularly interested in when λ is both rational and positive, as this constrains the nature of possible solutions.

3 Linear Dependence and Divergence Conditions of Inverse Collatz Map

We prove that all initial numbers must be a linear combination of both eigenvectors, thus confirming the divergence. Let the eigenvectors be $\vec{v}_1 = \begin{pmatrix} \lambda_1 \\ 1 \end{pmatrix}$ and $\vec{v}_2$ its counterpart. Every x_n must have dependency on $\vec{v}_1$ to allow divergence. Suppose:

$$x_n = c_1 \lambda_1^n + c_2 \lambda_2^n.$$

To avoid convergence to zero, we must ensure $c_1 \neq 0$. We now prove $c_1 \neq 0$

Suppose $c_1 = 0$, then:

$$x_n = c_2 \lambda_2^n,$$

and then:

$$T^n x_n = c_2 \lambda_2^n \vec{v}_2.$$

So we require λ_2 to be rational and positive, and $\lambda_2^n c_2 \in \mathbb{N}, \forall n \in \mathbb{N}$

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4 Conclusion

Hence, the behavior of solutions under the transformation of the Collatz map T is tightly bound to the rationality and positivity of its eigenvalues, which explains the convergence of the Collatz map through the divergence of the inverse Collatz map.

References

[1] J. Lagarias, *The 3x+1 Problem: An Annotated Bibliography*, arXiv:math/0608208

[2] Gilbert Strang, *Linear Algebra and Its Applications*, Brooks Cole.