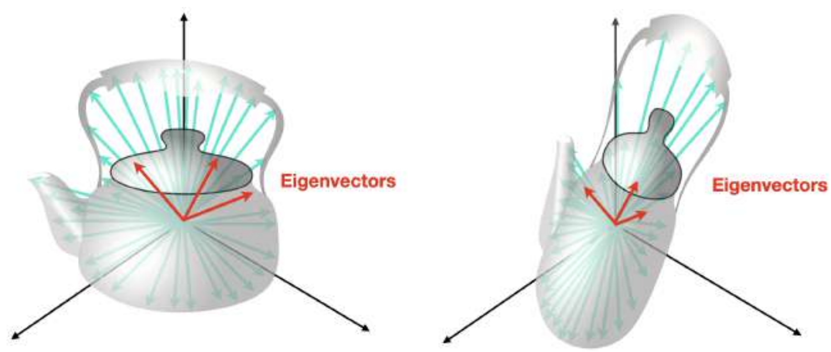
The Core of Eigenvalues & Eigenvectors

by DiBeos

Imagine a kettle such that each point at its surface is associated with a unique vector from the origin of our randomly chosen coordinate system. We want to transform this kettle such that all the green arrows can be scaled (increase or decrease in length) and can be rotated (change its angle with respect to the axes of our coordinate system). However, we can make a specific transformation where all the green arrows are free to move around and change as they want, deforming the surface of the kettle, but at the same time there are 3 special (red) vectors that do not rotate, and the only thing that they are allowed to do is to change their lengths (increase or decrease it). These special vectors are called *Eigenvectors* of the transformation, and the amount of which they increase or decrease is called their respective *Eigenvalues*.

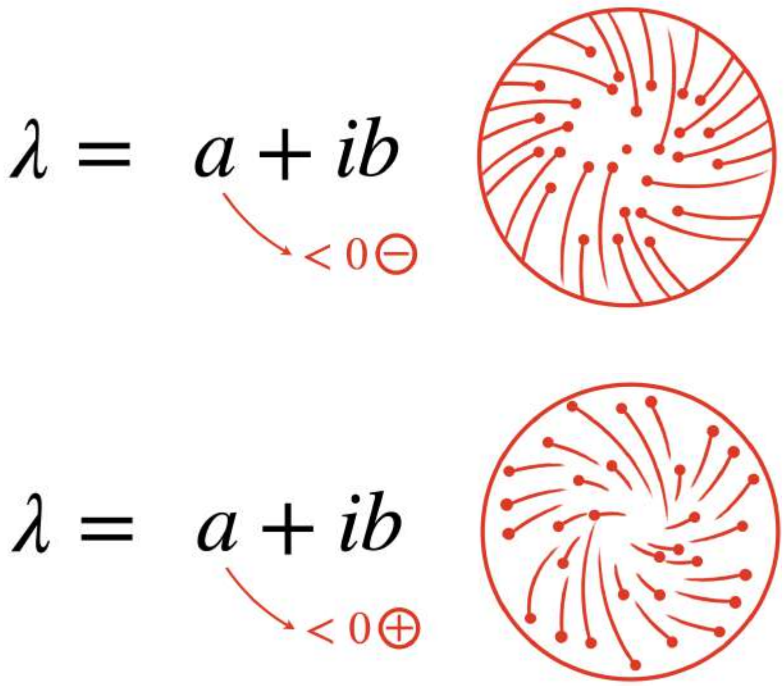


Not all transformations have these special eigenvectors, and therefore eigenvalues, but some do. When a transformation does have these properties we say that it is a **diagonalizable transformation.** We will see shortly what it is supposed to mean, but first of all let's see why these eigenvectors, with their respective eigenvalues, are so important.

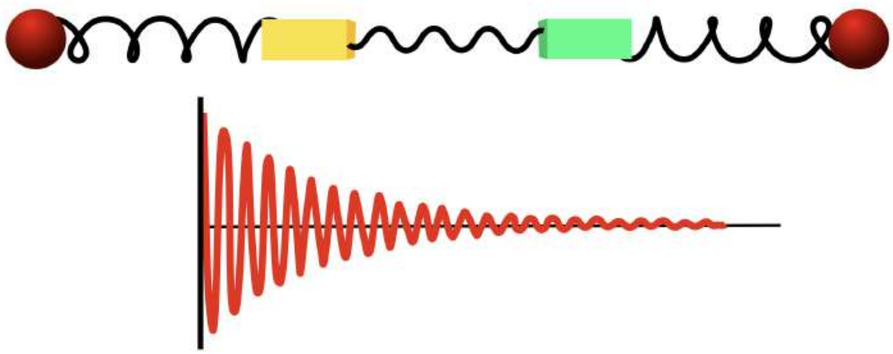
diagonalizable transformation

$$A = \begin{pmatrix} 4 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & 2 \end{pmatrix} \ \ldots \ D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$$

In **dynamical systems**, eigenvalues of the system's matrix indicate stability. For example, if all eigenvalues have negative real parts, the system converges to a stable state. Positive real parts suggest instability.

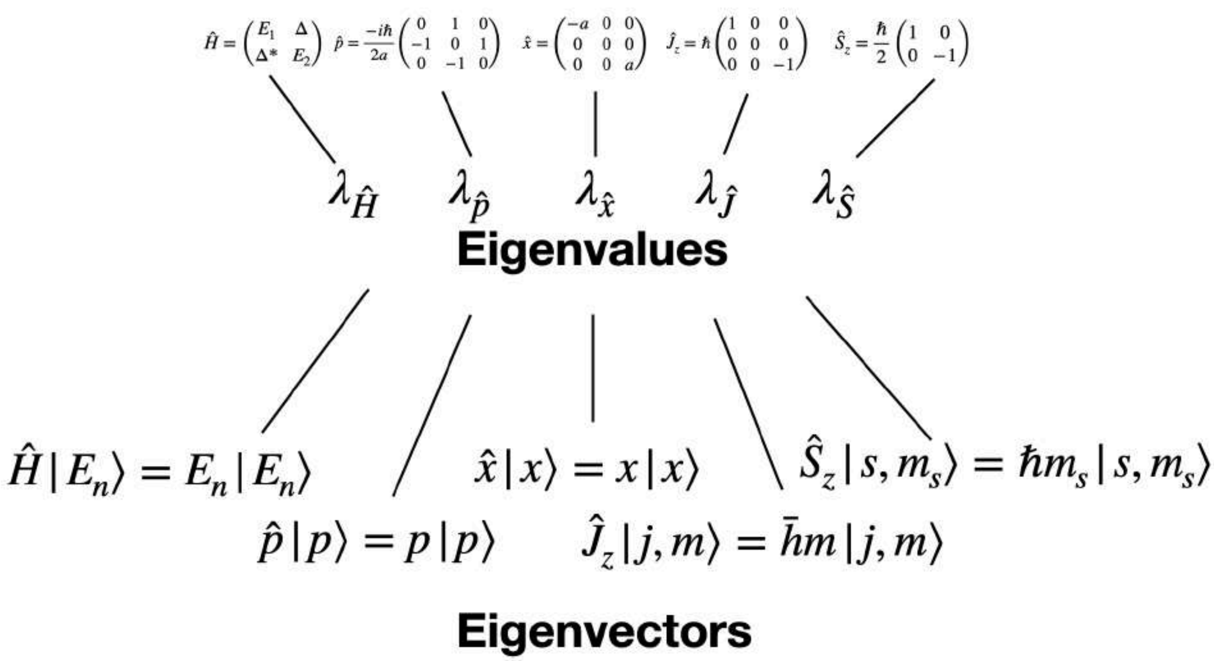


In **mechanical systems**, they relate to oscillations and damping behavior.



In **data science and machine learning**, eigenvectors of the covariance matrix identify the principal components of a dataset, which captures directions of maximum variance.

In **quantum mechanics**, observables (e.g. energy, momentum, position, angular momentum, spin, and so on) are all represented by operators (matrices). Their eigenvalues correspond to measurable quantities, and eigenvectors represent quantum states (e.g. the energy eigenstates, momentum eigenstates, spin eigenstates, and so on).

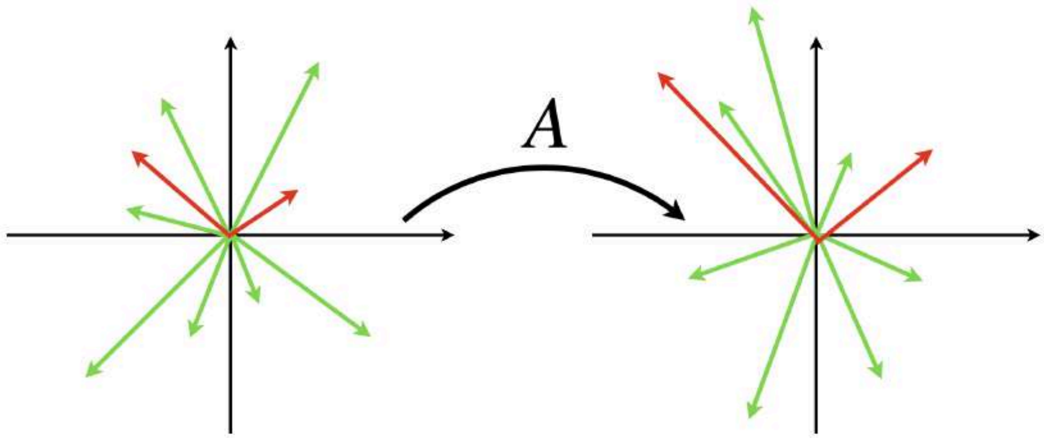


Eigenvalues can also correspond to natural frequencies of vibration in structures like bridges and buildings. Eigenvectors describe the mode shapes of these vibrations.

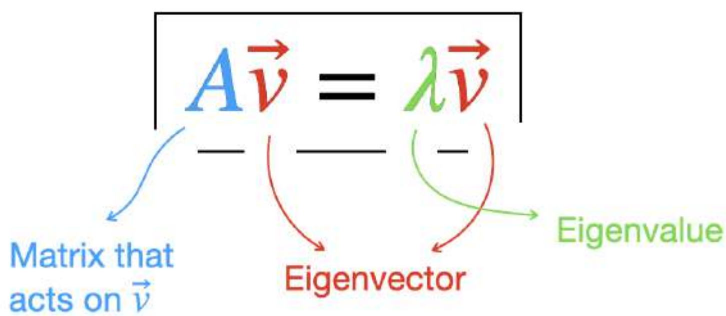
Ok, but how do we calculate these *special scalars* and *vectors*? Well, let's see a concrete example in 2 dimensions:

We have the transformation matrix $A = \begin{bmatrix} 1 & 3 \\ 2 & 1 \end{bmatrix}$.

It transforms vectors in the real plane into other vectors in the same real plane:



Since we want to calculate the “special” vectors that do not rotate, but just get scaled by a specific factor, we need the following equation to be satisfied:



Notice that this equation can be interpreted as: “after A acts on the vector $\vec{v}$, it does not change *direction*, just *intensity (length)* by a factor of λ”.

We can rewrite it as:

$$A\vec{v} = \lambda\vec{v} \Longrightarrow A\vec{v} - \lambda\vec{v} = \vec{0} \Longrightarrow \boxed{(A - \lambda I)\vec{v} = \vec{0}}$$

$$\vec{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(A - \lambda I)\vec{v} = \vec{0}$$

Now, we introduced the identity matrix I in front of $\vec{v}$ – acting on $\vec{v}$. We had to do it in order to make sure that we are comparing “apples with apples”, i.e. that we have a vector quantity in the RHS and another vector quantity in the LHS of the equation.

$$\underbrace{\overbrace{(A - \lambda I)}^{\text{matrix}}\vec{v}}_{\text{vector}} = \underbrace{\vec{0}}_{\text{vector}}$$

The equation above is actually a *homogeneous linear system*, and we will see why shortly. We want to find non-trivial solutions for this linear system. So, solutions that are non-zero vectors.

$$\vec{v} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If these non-trivial solutions exist, we say that the matrix $(A - \lambda I)$ is *singular*, and therefore:

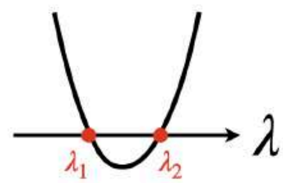
$$\underbrace{\det(A - \lambda I) = 0}_{\text{characteristic equation}}$$

This is called the *characteristic equation* and it will give us the eigenvalues that we are looking for!

$$\det(A - \lambda I) = 0 \Longrightarrow \det\left(\begin{bmatrix} 1 & +3 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) = 0$$

$$\Longrightarrow \det \begin{bmatrix} 1 - \lambda & 3 \\ 2 & 1 - \lambda \end{bmatrix} = 0 \Longrightarrow (1 - \lambda)^2 - 6 = 0$$

$$\Longrightarrow \lambda^2 - 2\lambda - 5 = 0$$



$$\lambda_{1,2} = \frac{-b \pm \overbrace{\sqrt{b^2 - 4ac}}^{\Delta}}{2a} = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot (-5)}}{2} =$$

$$= \begin{cases} = 1 + \sqrt{6} = \lambda_1 \\ = 1 - \sqrt{6} = \lambda_2 \end{cases}$$

Eigenvalues

We found the eigenvalues!

Now, let’s go back to the equation we saw before in order to calculate the eigenvectors related to each eigenvalue λ_1 and λ_2 .

$$(A-\lambda I)\vec{v}=\vec{0} \Longrightarrow \left(\begin{bmatrix}1 & 3\\ 2 & 1\end{bmatrix}-\begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)\begin{bmatrix}v_x\\ v_y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$$

$$\Longrightarrow \begin{bmatrix}1-\lambda & 3\\ 2 & 1-\lambda\end{bmatrix}\cdot\begin{bmatrix}v_x\\ v_y\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix} \Longrightarrow$$

$$\Longrightarrow \begin{cases}(1-\lambda)\,v_x+3v_y=0\\ 2v_x+(1-\lambda)\,v_y=0\end{cases}$$

Let’s find the eigenvector $\vec{v}_1$ of the eigenvalue $\lambda_1 = 1+\sqrt{6}$:

$$\begin{cases}(1-\lambda)\,v_x+3v_y=0\\ 2v_x+(1-\lambda)\,v_y=0\end{cases} \Longrightarrow \begin{cases}\left[1-\left(1+\sqrt{6}\right)\right]\,v_x+3v_y=0\\ 2v_x+\left[1-\left(1+\sqrt{6}\right)\right]\,v_y=0\end{cases}$$

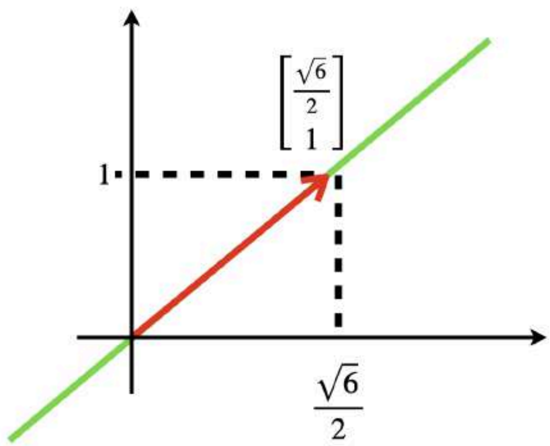
$$\Longrightarrow \begin{cases}-\sqrt{6}v_x+3v_y=0 \Longrightarrow v_x=\dfrac{3\cdot\sqrt{6}}{\sqrt{6}\cdot\sqrt{6}}v_y=\dfrac{3\sqrt{6}}{6}v_y\\ 2v_x-\sqrt{6}v_y=0\end{cases}$$

$$\boxed{v_x=\frac{\sqrt{6}}{2}v_y}$$

We found the same expression of the x-component in terms of its y-component in both equations. Now, we can build the general vector

$$\begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{2} v_y \\ v_y \end{bmatrix} = v_y \begin{bmatrix} \frac{\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

which *spans* the $1D$ space that is a line defined by the direction $\begin{bmatrix} \frac{\sqrt{6}}{2} \\ 1 \end{bmatrix}$, for all values $v_y \in \mathbb{R}$.

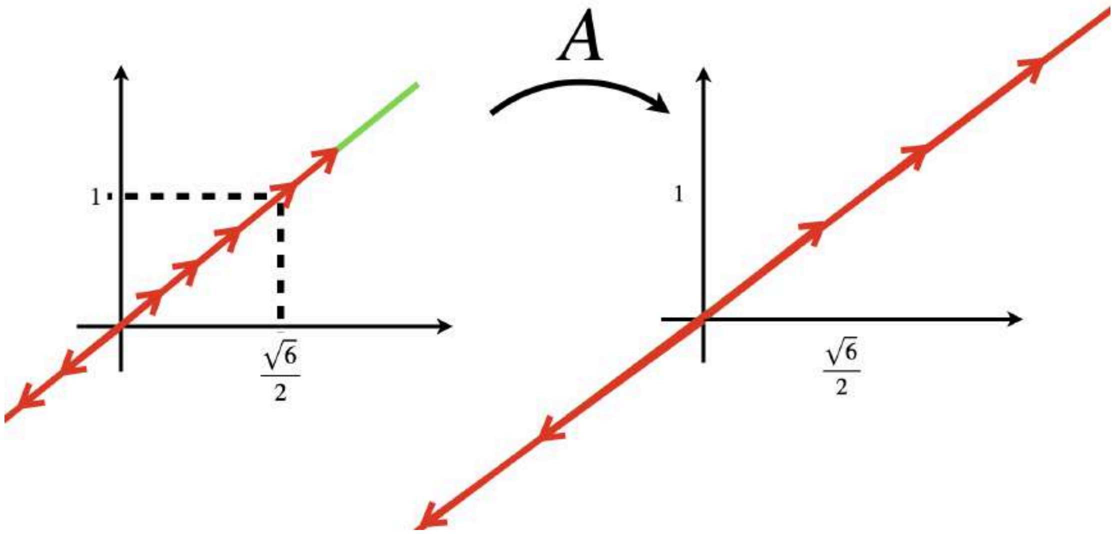


$$\lambda_1 = 1 + \sqrt{6} \implies \vec{v}_1 = \begin{bmatrix} \frac{\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

Therefore, we can consider $v_y = 1$, and as a consequence the eigenvector for the eigenvalue $\lambda_1 = 1 + \sqrt{6}$ is

$$\vec{v}_1 = \begin{bmatrix} \frac{\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

So, after the transformation A , all the vectors in the line defined by the direction $\begin{bmatrix} \frac{\sqrt{6}}{2} \\ 1 \end{bmatrix}$ were scaled by a factor of $1 + \sqrt{6} \approx 1.22$:



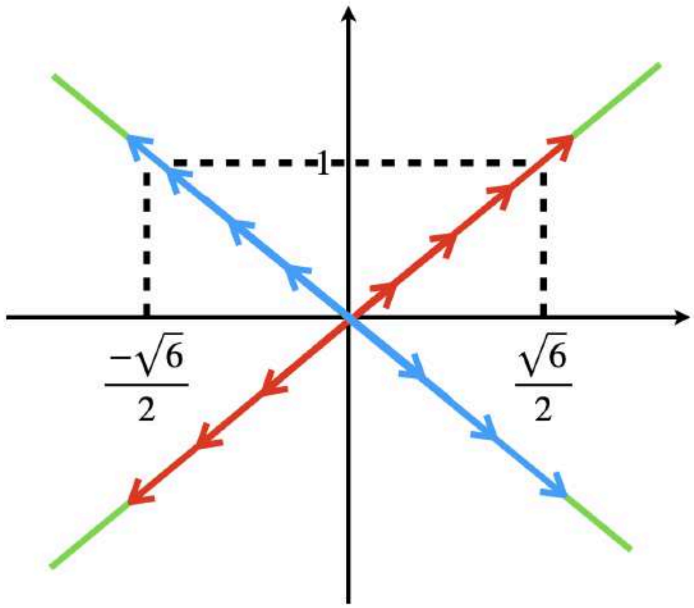
What about the second eigenvector? The one for $\lambda_2 = 1 - \sqrt{6}$?

Using the same process you can find $\vec{v}_2$:

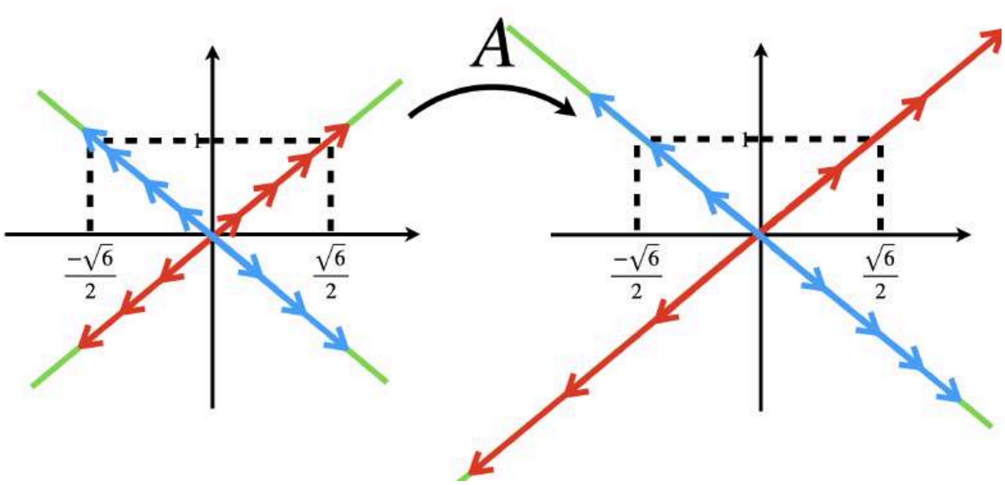
$$\lambda_1 = 1 + \sqrt{6} \implies \vec{v}_1 = \begin{bmatrix} \frac{\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

$$\lambda_2 = 1 - \sqrt{6} \implies \vec{v}_2 = \begin{bmatrix} \frac{-\sqrt{6}}{2} \\ 1 \end{bmatrix}$$

So, we found out that there are 2 linear spaces that get stretched in the transformation.



Awesome! But to be honest this is a very simple case, in which everything just turned out to be perfect. I invite you guys to invent a random square matrix and try to calculate its eigenvalues and eigenvectors using the same method. It is possible that you will run into some difficulties. But anyway, by the end of this document you will find 3 solved exercises to help you practice.



Let's see, for example, what happens when we change this transformation just a little bit. Replace 3 with $-\ 3$:

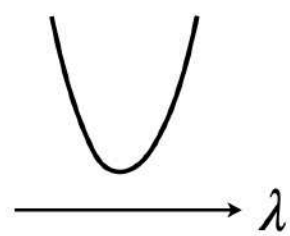
$$A = \begin{bmatrix} 1 & +3 \\ 2 & 1 \end{bmatrix} \quad A = \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}$$

Once again, we use the characteristic equation to find the eigenvalues:

$$\det(A - \lambda I) = 0 \implies \begin{bmatrix} 1-\lambda & -3 \\ 2 & 1-\lambda \end{bmatrix} = 0$$

$$\implies (1-\lambda)^2 + 6 = 0$$

$$\implies \boxed{\lambda^2 - 2\lambda + 7 = 0}$$



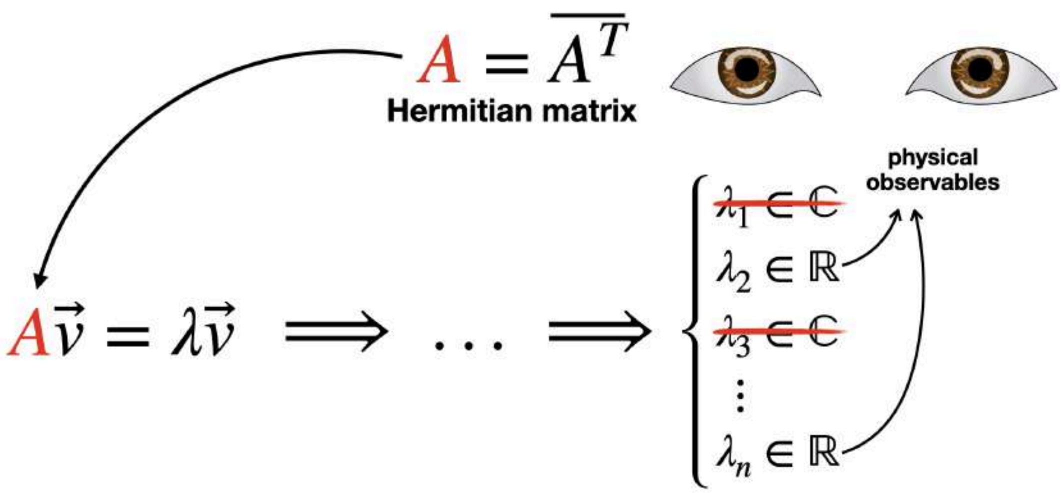
$$\Delta = b^2 - 4ac = 4 - 28 = -\ 24 < 0$$

The eigenvalues, then, would involve the term

$$\sqrt{\Delta} = \sqrt{-\ 24} = 2i\sqrt{6} \qquad \left(i := \sqrt{-\ 1}\right),$$

and therefore the eigenvalues will not be real numbers.

This is very important to notice, because, depending on the context, real eigenvalues can mean, and imply, completely different results from complex eigenvalues. For example, when applying it to the context of quantum mechanics (as you can see, that's our favorite application of linear algebra, haha), things that are considered as *physical observables* (like energy, position, momentum, and so on) are all eigenvalues of special matrix transformations called *Hermitian matrices (or operators)* that act on eigenvectors called *quantum states*, or *eigenstates*. If the eigenvalues are not real, then they cannot be interpreted as observable quantities in a physical experiment.



Going back to the mathematical theory of Eigenvalues and Eigenvectors, let's recap the general process for computing these quantities, and let's see some new important properties.

First step (1): Solve the characteristic equation

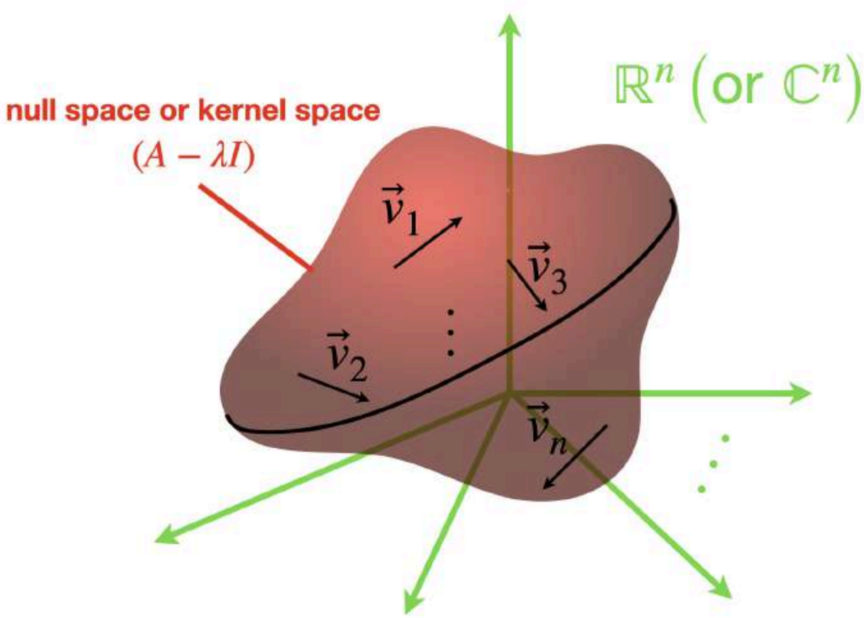
$$\det(A - \lambda I) = 0$$

and find the eigenvalues λ .

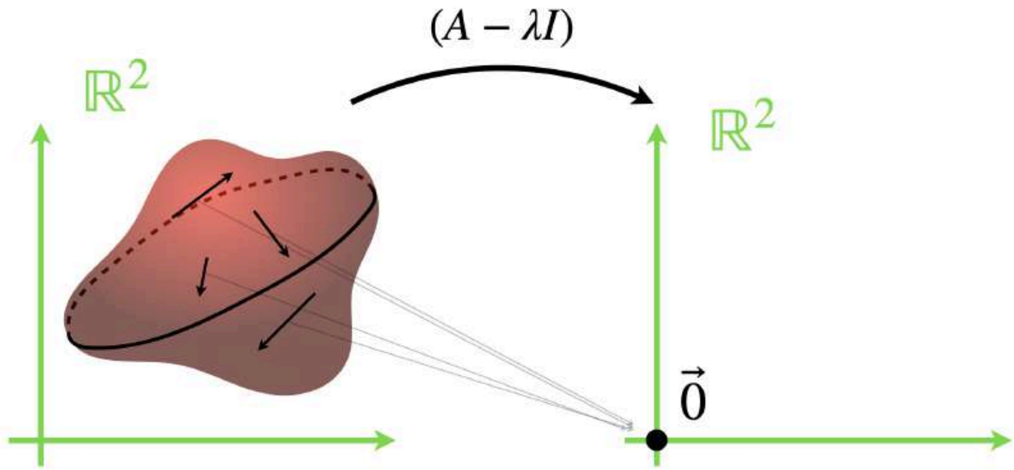
Second step (2): For each eigenvalue λ , solve the linear system of equations

$$
\begin{cases}
\lambda_1 \implies (A - \lambda_1 I)\,\vec{v}_1 = \vec{0} \implies \vec{v}_1 \\
\lambda_2 \implies (A - \lambda_2 I)\,\vec{v}_2 = \vec{0} \implies \vec{v}_2 \\
\lambda_3 \implies (A - \lambda_3 I)\,\vec{v}_3 = \vec{0} \implies \vec{v}_3 \\
\vdots \\
\lambda_n \implies (A - \lambda_n I)\,\vec{v}_n = \vec{0} \implies \vec{v}_n
\end{cases}
\quad \text{eigenvectors}
$$

to find the eigenvectors $\vec{v}$ of the matrix A . These eigenvectors live in a subspace of $\mathbb{R}^n$ (or of $\mathbb{C}^n$, if we are studying the complex case). This subspace is called the *null space* or *kernel* of the matrix $(A - \lambda I)$.



The kernel can also be seen as the space that has all its vectors mapped to the *null vector* $\left(\vec{0}\right)$ by the mapping $(A - \lambda I)$:



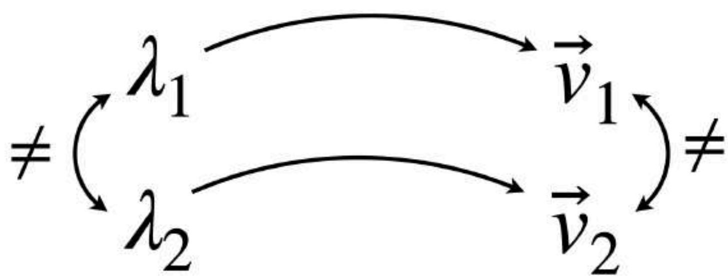
It's important to notice as well that we are allowed to call the kernel a subspace – it is more than a set with no structure – because of the following 3 properties that it possesses:

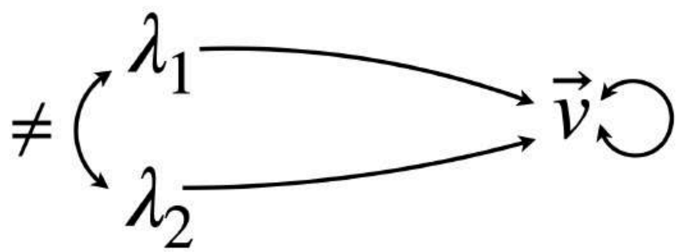
1. $Ker(A - \lambda I) \ni \vec{0}$;

2. $\vec{v}_1, \vec{v}_2 \in Ker(A - \lambda I) \quad \Rightarrow \quad \vec{v}_1 + \vec{v}_2 \in Ker(A - \lambda I)$
   (closed under addition) ;

3. $\vec{v} \in Ker(A - \lambda I) \ \wedge \ c \in \mathbb{R} \quad \Rightarrow \quad c\,\vec{v} \in Ker(A - \lambda I)$
   (closed under scalar multiplication)

Another important thing to notice is that (sometimes) different eigenvalues have different eigenvectors. However, there are instances in which 2 eigenvalues share the same eigenvector.





For example, let’s use our recipe to find eigenvalues and eigenvectors of the matrix transformation A:

$$A = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \qquad \det(A - \lambda I) = 0$$

$$\det(A - \lambda I) = 0 \implies \begin{bmatrix} 4-\lambda & 1 \\ 0 & 4-\lambda \end{bmatrix} = 0$$

$$\implies (4-\lambda)^2 = 0$$

(eigenvalue with multiplicity 2)

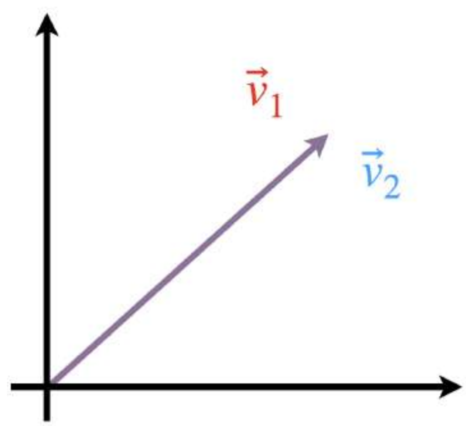
$$\implies \lambda_{1,2} = 4 \qquad \lambda_1 = 4 \qquad \lambda_2 = 4$$

We conclude that the matrix A is *not diagonalizable*, and that the 2 eigenvectors are *linearly dependent*. Therefore (for all practical purposes), these 2 eigenvectors are the same:

$$(A - 4I)\vec{v} = \vec{0} \implies \begin{bmatrix} 4-4 & 1 \\ 0 & 4-4 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

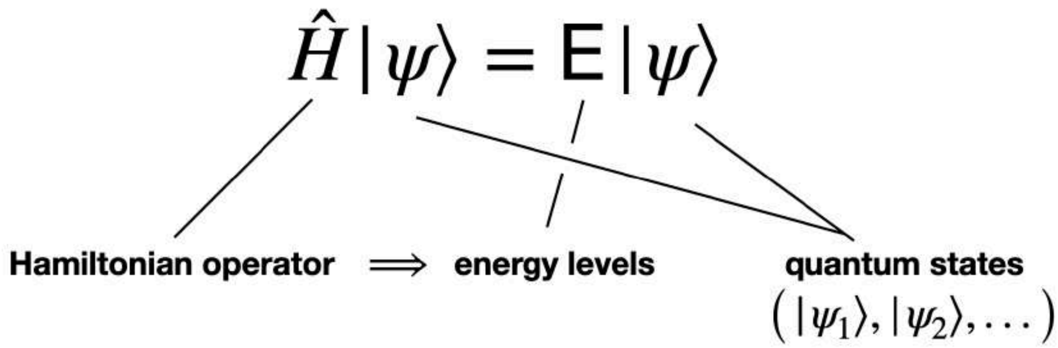
$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \implies v_y = 0$$

$$\implies \exists \text{ only 1 free variable } (v_x) : \boxed{\vec{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$



This kind of matrix is also called a *defective matrix*.

Degeneracy is very interesting for practical purposes as well. In quantum mechanics, we can look for eigenvalues and eigenvectors for the Schrödinger equation:



$\widehat{H}$ is the Hamiltonian operator (so, a matrix) which gives the *energy levels* of the system, represented by its eigenvalues. Its eigenvectors are *quantum states* (or *configurations*) $|\psi_1>$, $|\psi_2>$, … , of the system.

Degeneracy in this equation would mean that multiple quantum states $|\psi_1>$, $|\psi_2>$, … , correspond to the same energy E , which means that a system (like an electron in an atom, for example) can exist in different configurations while still having the same total energy.

In order to practice calculating eigenvalues and eigenvectors let's see 2 exercises:

P.S.: We have a website where we post all of the PDFs and other materials related to the content of the DiBeos channel, as well as a research section where you guys can send us your personal research, or just interesting explanations of math principles, so that we can post in the website and let it available for others to peer review your work and comment on it.

Exercise 1:

$$A = \begin{bmatrix} 5 & 2 \\ 2 & 4 \end{bmatrix}$$

Your Task:

1. **Find the eigenvalues** λ by solving $\det(A - \lambda I) = 0$.
2. **Find the eigenvectors** by solving $(A - \lambda I)v = 0$ for each λ.

Solution:

$$\boxed{\det(A - \lambda I) = 0} \implies \det\begin{pmatrix} 5-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} = 0 \implies$$

$$\implies 20 - 9\lambda + \lambda^2 - 4 = 0 \implies \boxed{\lambda^2 - 9\lambda + 16 = 0}$$

$$\lambda_{1,2} = \frac{9 \pm \sqrt{81-64}}{2} = \frac{9 \pm \sqrt{17}}{2} \text{ (eigenvalues)}$$

$$\boxed{(A - \lambda I)\vec{v} = \vec{0}} \implies \begin{pmatrix} 5-\lambda & 2 \\ 2 & 4-\lambda \end{pmatrix} \cdot \begin{pmatrix} v_x \\ v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies$$

$$\implies \begin{pmatrix} (5-\lambda)\cdot v_x + 2\cdot v_y \\ 2\cdot v_x + (4-\lambda)\cdot v_y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \implies \begin{cases} (5-\lambda)\cdot v_x + 2\cdot v_y = 0 \\ 2\cdot v_x + (4-\lambda)\cdot v_y = 0 \end{cases}$$

For $\lambda_{1,2} = \dfrac{9 \pm \sqrt{17}}{2}$:

$$\implies \begin{cases} \left(5 - \frac{9 \pm \sqrt{17}}{2}\right) v_x + 2v_y = 0 \\ 2v_x + \left(4 - \frac{9 \pm \sqrt{17}}{2}\right) v_y = 0 \end{cases} \implies \begin{cases} \frac{1 \pm \sqrt{17}}{2} v_x + 2v_y = 0 \\ 2v_x + \frac{-1 \mp \sqrt{17}}{2} v_y = 0 \end{cases}$$

$$\Longrightarrow \quad v_x = \frac{-2}{\left(\frac{1 \pm \sqrt{17}}{2}\right)} v_y \Longrightarrow \quad v_x = \frac{-4}{1 \pm \sqrt{17}} v_y \frac{\cdot (1 \mp \sqrt{17})}{\cdot (1 \mp \sqrt{17})} \Longrightarrow$$

$$\Longrightarrow \quad v_x = \frac{\overset{1}{\cancel{-4}} \cdot \left(1 \mp \sqrt{17}\right)}{-\cancel{16}\ 4} v_y \Longrightarrow \quad \boxed{v_x = \frac{1 \pm \sqrt{17}}{4} v_y}$$

$$\xRightarrow{\left(v_y \equiv 1\right)} \quad \vec{v}_{1,2} = \begin{pmatrix} \frac{1 \pm \sqrt{17}}{4} \\ 1 \end{pmatrix} \text{ (eigenvectors)}$$

Exercise 2:

$$A = \begin{bmatrix} 0 & -1 & 2 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix}$$

Your Task:

1. **Find the eigenvalues** λ by solving $\det(A - \lambda I) = 0$.
2. **Find the eigenvectors** by solving $(A - \lambda I)v = 0$ for each λ.

Solution:

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 4 & 1 \\ 2 & 1 & 3 \end{bmatrix}$$

$$\boxed{\det(A - \lambda I) = 0} \implies \det \begin{bmatrix} 3-\lambda & 1 & 2 \\ 1 & 4-\lambda & 1 \\ 2 & 1 & 3-\lambda \end{bmatrix} = 0 \implies$$

$$\implies \left[(3-\lambda)(4-\lambda)(3-\lambda) + 1 \cdot 1 \cdot 2 + 2 \cdot 1 \cdot 1\right] -$$

$$\left[2 \cdot 2 \cdot (4-\lambda) + 1 \cdot 1 \cdot (3-\lambda) + (3-\lambda) \cdot 1 \cdot 1\right] = 0 \implies$$

$$\implies (12 - 7\lambda + \lambda^2)(3-\lambda) + 4 - (16 - 4\lambda + 3 - \lambda + 3 - \lambda) = 0$$

$$\implies \boxed{-\lambda^3 + 10\lambda^2 - 27\lambda + 18 = 0} \iff \boxed{(\lambda - 6)(\lambda - 3)(\lambda - 1) = 0}$$

$\therefore \quad \lambda_1 = 3 \;;\; \lambda_2 = 6 \;;\; \lambda_3 = 1$ (eigenvalues)

$$\boxed{(A - \lambda I)\vec{v} = \vec{0}} \xRightarrow{(\lambda_1 = 3)} \begin{bmatrix} 3-3 & 1 & 2 \\ 1 & 4-3 & 1 \\ 2 & 1 & 3-3 \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \vec{0} \implies$$

$$\implies \begin{cases} v_y + 2v_z = 0 \implies \boxed{v_y = -2v_z} \\ v_x + v_y + v_z = 0 \implies v_x = -v_y - v_z \implies \boxed{v_x = v_z} \\ 2v_x + v_y = 0 \end{cases}$$

$$\therefore \quad \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix} = \begin{bmatrix} v_z \\ -2v_z \\ v_z \end{bmatrix} = v_z \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \implies \boxed{\vec{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}} \quad \text{(eigenvector)}$$

$$\boxed{(A-\lambda I)\,\vec{v}=\vec{0}}\xRightarrow{(\lambda_2=6)}\begin{bmatrix}3-6 & 1 & 2\\ 1 & 4-6 & 1\\ 2 & 1 & 3-6\end{bmatrix}\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}=\vec{0}\Longrightarrow$$

$$\Longrightarrow\begin{cases}-3v_x+v_y+2v_z=0 \Longrightarrow \boxed{v_y=3v_x-2v_z}\ (\text{I})\\ v_x-2v_y+v_z=0 \Longrightarrow v_x=2v_y-v_z=2\,(3v_x-2v_z)-v_z \Rightarrow \boxed{v_x=v_z}\ (\text{II})\\ 2v_x+v_y-3v_z=0\end{cases}$$

(II) in (I):

$$v_y=3v_x-2v_x \Longrightarrow \boxed{v_y=v_x}$$

$$\therefore\ \begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}=\begin{bmatrix}v_x\\ v_x\\ v_x\end{bmatrix}=v_x\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\Longrightarrow \boxed{\vec{v}_2=\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}}$$ (eigenvector)

$$\boxed{(A-\lambda I)\,\vec{v}=\vec{0}}\xRightarrow{(\lambda_3=1)}\begin{bmatrix}3-1 & 1 & 2\\ 1 & 4-1 & 1\\ 2 & 1 & 3-1\end{bmatrix}\begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}=\vec{0}\Longrightarrow$$

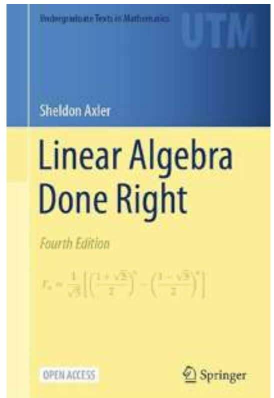
$$\Longrightarrow\begin{cases}2v_x+v_y+2v_z=0 \Longrightarrow \boxed{v_y=-2v_x-2v_z}\ (\text{I})\\ v_x+3v_y+v_z=0 \Longrightarrow v_x=-3\,(-2v_y-2v_z)-v_z \Rightarrow \boxed{v_x=-v_z}\ (\text{II})\\ 2v_x+v_y+2v_z=0\end{cases}$$

(II) in (I):

$$v_y=-2v_x+2v_x \Longrightarrow \boxed{v_y=0}$$

$$\therefore\ \begin{bmatrix}v_x\\ v_y\\ v_z\end{bmatrix}=\begin{bmatrix}-v_z\\ 0\\ v_z\end{bmatrix}=v_x\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}\Longrightarrow \boxed{\vec{v}_2=\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix}}$$ (eigenvector)

For more on eigenvectors and eigenvalues, check out the following excellent book:



https://amzn.to/3F3amMY

Important properties of eigenvalues and eigenvectors:

- *Number of eigenvectors:* An $n \times n$ matrix has at most n eigenvalues (real or complex , counting multiplicities).

- *Eigenvectors Correspond to Each Eigenvalue:* Different eigenvalues usually have different eigenvectors, but some eigenvalues can share eigenvectors.

- *Determinant and Trace Connection:*

  The determinant of A is the product of its eigenvalues:

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

The trace (sum of diagonal elements) is the sum of its eigenvalues:

$$\mathrm{Tr}(A) = \sum_{i=1}^{n} \lambda_i$$

- *Diagonalization Condition:* If A has n linearly independent eigenvectors, it can be diagonalized as:

$$A = PDP^{-1}$$

where D is a diagonal matrix of eigenvalues and P is a matrix whose columns are the eigenvectors.

- *Defective Matrices:* If a matrix does not have enough linearly independent eigenvectors, it is called defective and cannot be diagonalized.

Special Cases and Tricks:

- *Symmetric Matrices $A^T = A$:*

  Always have real eigenvalues.

  Have orthogonal eigenvectors (important in physics and engineering).

- *Skew-Symmetric Matrices $A^T = -A$:*

  Eigenvalues are either purely imaginary or zero.

- *Orthogonal Matrices* $A^T A = I$ *:*

  Eigenvalues have absolute value 1 (they lie on the unit circle in the complex plane).

- *Diagonal and Triangular Matrices:*

  The eigenvalues are just the diagonal elements.

- *Power of a Matrix and Eigenvalues:*

  If A has eigenvalues λ_1, λ_2, ..., then:

  $$A^k \text{ has eigenvalues } \lambda_1^k, \lambda_2^k, \ldots$$

- *Eigenvectors Can Be Scaled:*

  If $\vec{v}$ is an eigenvector of A, then any scaled version $c\vec{v}$ (where $c \neq 0$) is also an eigenvector.

Common Mistakes:

- *Forgetting to Check Linear Independence:*

  If two eigenvectors are not linearly independent, you cannot diagonalize the matrix.

- *Not Checking for Complex Eigenvalues:*

If the determinant gives a negative discriminant, expect complex eigenvalues and eigenvectors.

- *Singular Matrices Always Have a Zero Eigenvalue:*

  If $det(A) = 0$, then at least one eigenvalue must be zero.

- *Eigenvectors Are Not Always Unique:*

  If an eigenvalue has multiplicity greater than 1, it might have multiple independent eigenvectors (or not, in the defective case).

Please, if you find this document useful, let us know. Or if you found typos and things to improve, let us know as well. Your feedback is very important to us. We're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com