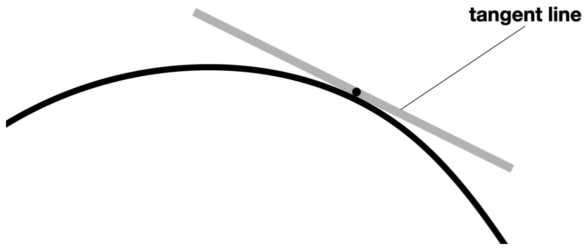
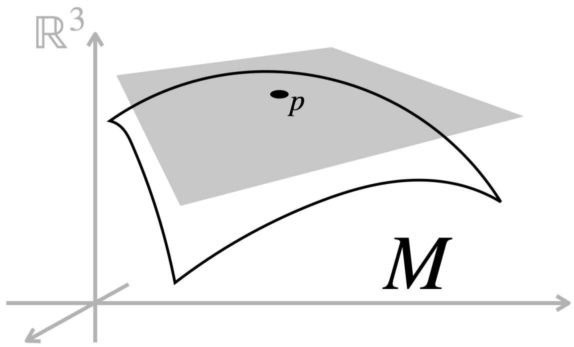
Tangent Space Basis

Think of a curve. We will locate this curve in a dimension which we can easily imagine, in $\mathbb{R}^2$. Now, draw a line which lightly touches the curve at just one specific point. This is a **tangent line**.



Now say that instead of a line, we draw a 2 dimensional space, much like a loosely hanging sheet. This is a **manifold**. It is located in a 3 dimensional space $\mathbb{R}^3$. Say we pick a point *p*, and want to draw something equivalent to a tangent line but in 2 dimensions. We would therefore draw a **tangent space**. It's labeled as *TpM* and literally means "the tangent space of the manifold M at the point p".



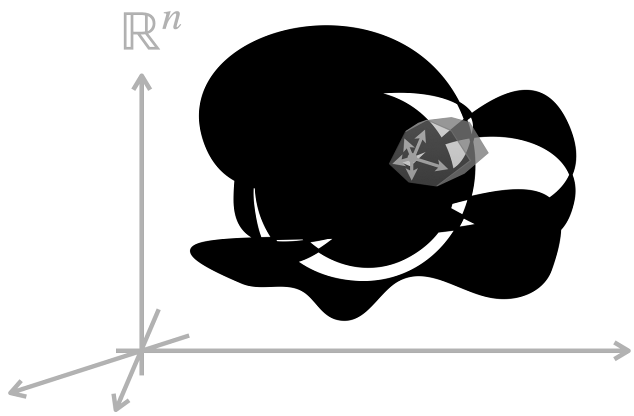
Mathematically, we would describe the tangent space just like how we describe locations in everyday space using coordinates, we describe directions in the tangent space using **coordinate systems** tied to the manifold.

Tangent spaces are abstract constructions that represent all possible directions in which one can move from a point on a manifold. This concept is known as a **vector**. Vectors are like "arrows" which have direction, and magnitude,

On a curved line, there are only two possible directions for vectors: forward and backward along the curve.

On a 2-dimensional tangent plane, there are infinitely many possible directions, forming a "circle of choices" or directions around the point *p*. These vectors tell us all the possible directions we can travel in the immediate neighborhood of *p*.

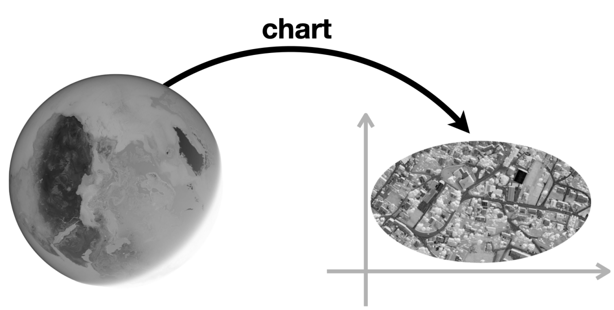
When we have a higher dimensional manifold embedded in a higher dimensional Euclidean space, these vector spaces become abstract, and impossible to visualize. These spaces still touch the manifold at one specific point.



But here's the catch: Manifolds don't actually need to be embedded in a higher dimensional space.

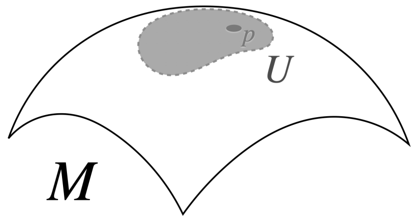
A manifold does not have to have a space surrounding it. It *is* the space, and there is nothing beyond it. That's why vectors can't poke out of it, because there's nothing to poke out into.

This is similar to zooming in on a globe of the Earth until it looks flat. That's what we're doing conceptually when we use **charts** to map part of a curved manifold onto Euclidean space, where we can perform calculus.

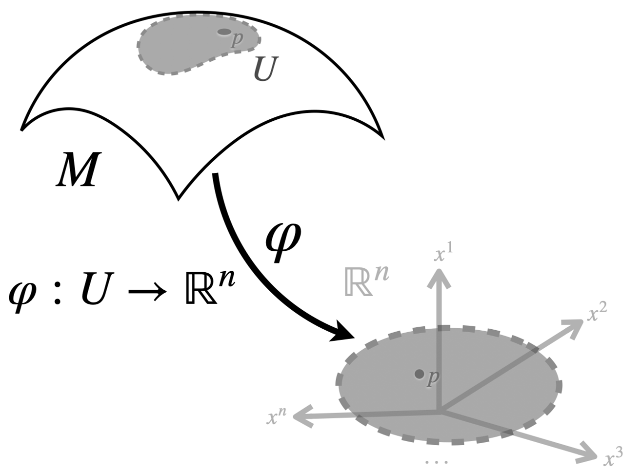


So the question becomes: how are we supposed to do that? And how would you map an entire tangent space? First, let's start with finding just one single vector.

We have a manifold *M*, and pick a point *p*. Since we are talking about a manifold in higher dimensions, it's pretty hard to pin-point the exact location of the *p*. So instead, we create a neighborhood of points where *p* floats freely, but does not exit the boundaries of the region, and label it *U*. It is formally called an **open subset** of *M*.

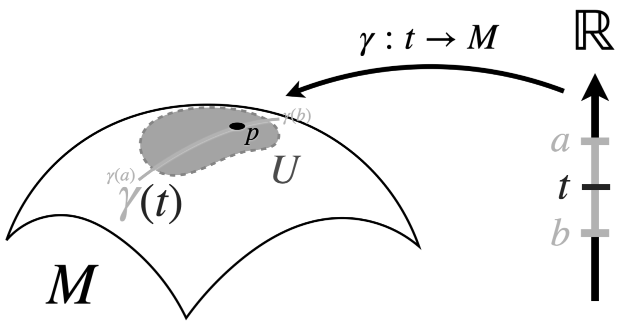


In order to perform any kind of calculus at all on *p* and on *U*, we have to map it to a Euclidean space $\mathbb{R}^n$, where calculus is well-defined. This is done using a chart φ.



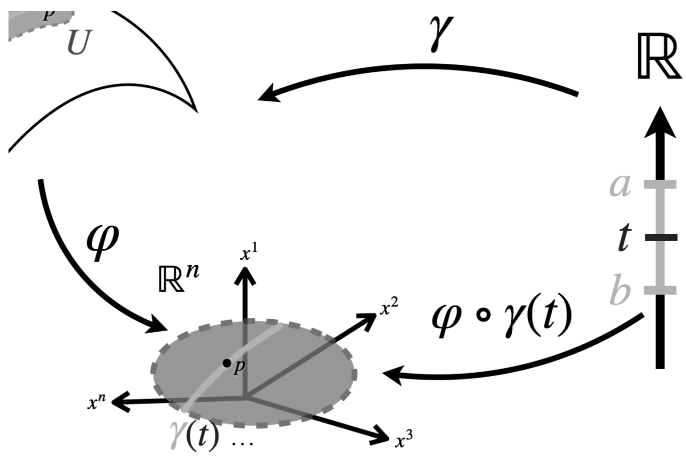
Say we have a curve γ passing through the manifold through the point *p*. Here, we also have to map the parameter of time, or *t*.

t which finds itself in another Euclidean space ℝ. The line itself is just one dimensional representing "time". *t* takes values in the interval *[a,b]* ⊆ ℝ .We therefore map it to M.



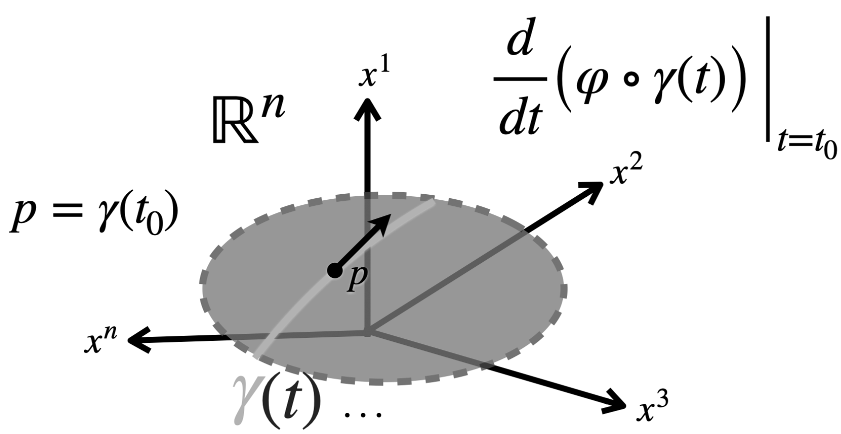
The parameter t tells us where we are on the curve at a specific time. For example, $\gamma(t_0) = p$ means that at time t_0, the curve passes through the point *p*.

To calculate the velocity of the curve at *p*, we need to work in a Euclidean space $\mathbb{R}^n$, where derivatives are well-defined. To do this we combine the chart φ and the curve *γ(t)*. The composition φ∘γ maps the curve on the manifold to a curve in $\mathbb{R}^n$.



Once the curve is mapped to Euclidean space, we can finally compute the velocity vector at point *p* by taking the derivative of $\varphi \circ \gamma(t)$ with respect to *t*.

This derivative gives the velocity vector of the curve at the point $p = \gamma(t_0)$, but now expressed in terms of the local coordinates.



All of this was to introduce tangent vectors as **velocities of curves** passing through a point p on the manifold *M*. But this is not enough for practical analysis because a single tangent vector is just one direction of movement at p, but instead we need to describe all possible directions at p, which form the **tangent space** T_pM. This is where the idea of tangent space basis comes in.

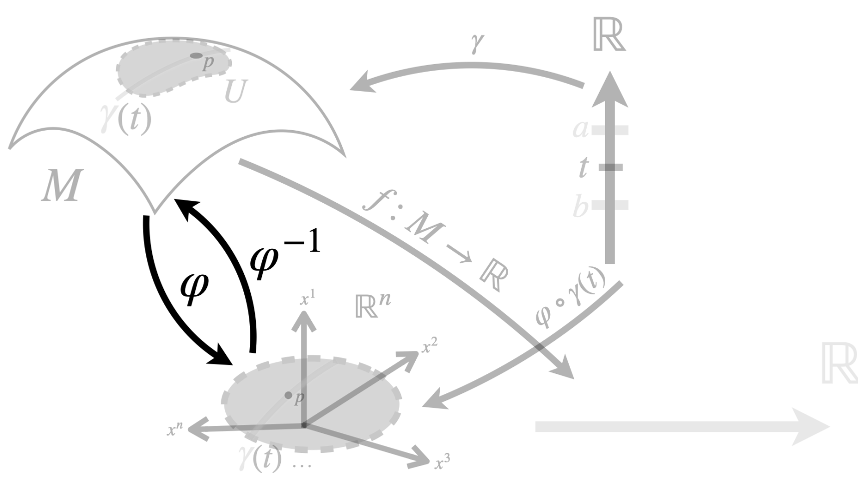
We need to introduce an arbitrary function $f : M \rightarrow \mathbb{R}$, an entirely different field called a **scalar field**. What it does is assign scalars to each point on *M*. Scalars are much like vectors, except they don’t have a direction. It’s much like velocity versus temperature, for example.

Say we want to find the rate of change of our function f as we walk along our curve $\gamma(t)$. We want to look at a new definition of "velocity" relative to this test function at our point *p* on the manifold.

$$\left.\frac{d f\circ\gamma(t)}{dt}\right|_{t=t_0}$$

But, here’s a really fun trick we can do, by introducing φ and its inverse. It’s a trick because, now, it doesn’t really do anything, it’s just like adding zero.

$$\left.\frac{d\,(f\circ\varphi^{-1}\circ\varphi\circ\gamma)(t)}{dt}\right|_{t\,=\,t_0}$$



Because this introduction is quote on quote harmless, we can therefore put it like this:

$$= \frac{d((f \circ \varphi^{-1}) \circ (\varphi \circ \gamma))(t)}{dt}\Big|_{t=t_0}$$

Now, we will introduce the chain rule:

$$= \sum_i \frac{\partial(f \circ \varphi^{-1})(x)}{\partial x_i}\Big|_{x=\varphi\circ\gamma(t_0)} \frac{d(\varphi \circ \gamma)^i(t)}{dt}\Big|_{t=t_0}$$

Which brings us here:

$$= \sum_i \frac{\partial(f \circ \varphi^{-1})(x)}{\partial x_i}\Big|_{x=\varphi(p)} \frac{d(\varphi \circ \gamma)^i(t)}{dt}\Big|_{t=t_0}$$

Since $p = \gamma(t_0)$. With the left part being the basis for component i, and the right part the "velocity" of component *i* with respect to φ.

We're going to rewrite the basis like this:

$$\left(\frac{\partial}{\partial x^i}\right)_p (f) := \frac{\partial(f \circ \varphi^{-1})(\varphi(p))}{\partial x_i}$$

φ is specified by x^i. The f was only needed to define the basis of the tangent space. It was arbitrary, and we can get rid of it now.

So for every tangent vector $v \in TpM$, we have this:

$$\vec{v} = \sum_{i=1}^{n} u(x^i) \cdot \left(\frac{\partial}{\partial x^i}\right)_p$$

$$= \sum_{i=1}^{n} \frac{d(\phi \circ \gamma)^i(t)}{dt}\Bigg|_{t=t_0} \cdot \left(\frac{\partial}{\partial x^i}\right)_p$$

The basis is actually a set of differential operators, not the actual vectors on the test function f, which make up a vector space.

tangent space