$A \subseteq \mathbb{R} \quad \wedge \quad \lambda \in \mathbb{R}$:

$(a) \vdash sup(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda > 0$

$(b) \vdash inf(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda > 0$

$(c) \vdash inf(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda < 0$

$(d) \vdash sup(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda < 0$

$A \subseteq \mathbb{R} \quad \wedge \quad \lambda \in \mathbb{R}$:

$(a) \vdash sup(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda > 0$

Definitions:

1. $sup(A)$ is the minimum upper bound of A ;
2. Upper bound of A is any element $M \in \mathbb{R} : M \geq x \ , \forall x \in A$;
3. $inf(A)$ is the maximum lower bound of A ;
4. Lower bound of A is any element $m \in \mathbb{R} : m \leq x \ , \forall x \in A$;
5. $\lambda A := \{\lambda \cdot x : x \in A\}$

$A \subseteq \mathbb{R} \;\wedge\; \lambda \in \mathbb{R}$: *Ad absurdum…*

$(a) \vdash sup(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda > 0$. Proof:

<u>Thesis statement</u>: $\boxed{sup(\lambda A) > \lambda\, sup(A)}^{\bigstar}$ *for* $\lambda > 0$

$sup(\lambda A)$ is the smallest u. b. (upper bound) of λA , but in $\bigstar$ we just found an <u>element</u> ($\lambda\, sup(A)$) in $\mathbb{R}$ that is less than $sup(\lambda A)$

$$\Downarrow$$

$\lambda\, sup(A)$ is <u>not</u> an u. b. of λA

$\bigstar \; \boxed{sup(\lambda A) > \lambda\, sup(A)}$ *for* $\lambda > 0$

$\lambda\, sup(A)$ is <u>not</u> an u. b. of $\lambda A \implies \exists x_\lambda \in \lambda A : \boxed{\lambda\, sup(A) < x_\lambda}$ ⬢

But… $x_\lambda \in \lambda A \implies x_\lambda := \lambda x$, for some $x \in A$

⬢ $\implies \cancel{\lambda}\, sup(A) < \cancel{\lambda} x \implies sup(A) < x$, for some $x \in A \implies$

$\implies \exists x \in A : \; sup(A) < x \implies sup(A)$ is <u>not</u> an u. b. of A ↯

$A \subseteq \mathbb{R} \ \wedge \ \lambda \in \mathbb{R}$: *Ad absurdum…*

(a) $\vdash sup(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda > 0$. Proof:

Thesis statement: ~~$\boxed{sup(\lambda A) > \lambda\, sup(A)}$~~ $for \quad \lambda > 0$ ★

upper bound

$sup(\lambda A)$ is the smallest u. b. of λA , but in ★ we just found an element in $\mathbb{R}$ that is less than $sup(\lambda A)$

$\lambda\, sup(A)$

$$\Downarrow$$

$\lambda\, sup(A)$ is not an u. b. of λA

★ ~~$\boxed{sup(\lambda A) > \lambda\, sup(A)}$~~ $for \quad \lambda > 0$

Thesis statement: $\boxed{sup(\lambda A) < \lambda\, sup(A)}$ $for \quad \lambda > 0 \quad \Longrightarrow$

$$\Longrightarrow \quad \frac{sup(\lambda A)}{\lambda} < sup(A) \quad for \quad \lambda > 0$$

$sup(A)$ is the smallest u. b. of A , but we just found an element in $\mathbb{R}$ that is less than $sup(A)$ $\Longrightarrow$ $\dfrac{sup(\lambda A)}{\lambda}$ is not an u. b. of A

(BTW, consider becoming a member of the channel!) Thanks!

★ ~~$\boxed{sup(\lambda A) > \lambda\, sup(A)}$~~ $for \quad \lambda > 0$

$\dfrac{sup(\lambda A)}{\lambda}$ is <u>not</u> an u. b. of A $\quad \Longrightarrow \quad \exists\, x \in A : \dfrac{sup(\lambda A)}{\lambda} < x \quad \Longrightarrow$

$\Longrightarrow \quad \exists\, x \in A : \quad sup(\lambda A) < \lambda x \quad \Longrightarrow \quad \exists\, x_\lambda := \lambda x \in \lambda A : \quad sup(\lambda A) < x_\lambda$

$\Longrightarrow \quad sup(\lambda A)$ is <u>not</u> an u. b. of λA ↯

★ ~~$\boxed{sup(\lambda A) > \lambda\, sup(A)}$~~ $for \quad \lambda > 0$

<u>Thesis statement</u>: ~~$\boxed{sup(\lambda A) < \lambda\, sup(A)}$~~ $for \quad \lambda > 0 \qquad \Longrightarrow$

$\Longrightarrow \quad \dfrac{sup(\lambda A)}{\lambda} < sup(A) \quad for \quad \lambda > 0$

$sup(A)$ is the smallest u. b. of A , but we just found an element in $\mathbb{R}$ that is less than $sup(A)$ $\quad \Longrightarrow \quad \dfrac{sup(\lambda A)}{\lambda}$ is <u>not</u> an u. b. of A

💎 $\boxed{sup(\lambda A) < \lambda\, sup(A)}$ $\textit{for} \quad \lambda > 0$ ★ $\boxed{sup(\lambda A) > \lambda\, sup(A)}$ $\textit{for} \quad \lambda > 0$

If neither ★ nor 💎 are true, then:

$$\boxed{sup(\lambda A) = \lambda\, sup(A)} \quad \textit{for} \quad \lambda > 0$$

$\blacksquare$

$$A \subseteq \mathbb{R} \quad \land \quad \lambda \in \mathbb{R} \;:$$

$$(a) \vdash sup(\lambda A) = \lambda\, sup(A) \quad \textit{if} \quad \lambda > 0$$

$$(b) \vdash inf(\lambda A) = \lambda\, inf(A) \quad \textit{if} \quad \lambda > 0$$

$$(c) \vdash inf(\lambda A) = \lambda\, sup(A) \quad \textit{if} \quad \lambda < 0$$

$$(d) \vdash sup(\lambda A) = \lambda\, inf(A) \quad \textit{if} \quad \lambda < 0$$

$A \subseteq \mathbb{R} \;\wedge\; \lambda \in \mathbb{R}$: *Ad absurdum…*

$(b) \vdash inf(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda > 0$. Proof:

Thesis statement: $\boxed{inf(\lambda A) < \lambda\, inf(A)}$ *for* $\lambda > 0$

lower bound

$inf(\lambda A)$ is the greatest l. b. of λA , but in we just found an element in $\mathbb{R}$ that is greater than $inf(\lambda A)$

$\lambda\, inf(A)$

$$\Downarrow$$

$\lambda\, inf(A)$ is not a l. b. of λA

$\boxed{inf(\lambda A) < \lambda\, inf(A)}$ *for* $\lambda > 0$

$\lambda\, inf(A)$ is not a l. b. of $\lambda A \implies \exists x_\lambda \in \lambda A : \boxed{\lambda\, inf(A) > x_\lambda}$

But… $x_\lambda \in \lambda A \implies x_\lambda := \lambda x$, for some $x \in A$

$\implies \cancel{\lambda}\, inf(A) > \cancel{\lambda} x \implies inf(A) > x$, for some $x \in A \implies$

$\implies \exists x \in A : \; inf(A) > x \implies inf(A)$ is not a l. b. of A ↯

$A \subseteq \mathbb{R} \;\; \wedge \;\; \lambda \in \mathbb{R}$: *Ad absurdum…*

$(b) \vdash inf(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda > 0$. Proof:

<u>Thesis statement</u>: ~~$\boxed{inf(\lambda A) < \lambda\, inf(A)}$~~ $for \quad \lambda > 0$

(lower bound)

$inf(\lambda A)$ is the greatest l. b. of λA , but in we just found an <u>element</u> in $\mathbb{R}$ that is greater than $inf(\lambda A)$

($\lambda\, inf(A)$)

$$\Downarrow$$

$\lambda\, inf(A)$ is <u>not</u> a l. b. of λA

~~$\boxed{inf(\lambda A) < \lambda\, inf(A)}$~~ $for \quad \lambda > 0$

<u>Thesis statement</u>: $\boxed{inf(\lambda A) > \lambda\, inf(A)}$ $for \quad \lambda > 0 \quad \Longrightarrow$

$$\Longrightarrow \quad \frac{\text{inf}(\lambda A)}{\lambda} > inf(A) \quad for \quad \lambda > 0$$

$inf(A)$ is the greatest l. b. of A , but we just found an element in $\mathbb{R}$ that is greater than $inf(A)$ $\quad \Longrightarrow \quad \dfrac{inf(\lambda A)}{\lambda}$ is <u>not</u> a l. b. of A

~~$\boxed{inf(\lambda A) < \lambda\, inf(A)}$~~ $for \quad \lambda > 0$

$$\frac{inf(\lambda A)}{\lambda} \text{ is } \underline{\text{not}} \text{ a l. b. of } A \implies \exists\, x \in A : \; \frac{inf(\lambda A)}{\lambda} > x \implies$$

$$\implies \exists\, x \in A : \; inf(\lambda A) > \lambda x \implies \exists\, x_\lambda := \lambda x \in \lambda A : \; inf(\lambda A) > x_\lambda$$

$$\implies inf(\lambda A) \text{ is } \underline{\text{not}} \text{ a l. b. of } \lambda A$$ ↯

~~$\boxed{inf(\lambda A) < \lambda\, inf(A)}$~~ $for \quad \lambda > 0$

<u>Thesis statement</u>: ~~$\boxed{inf(\lambda A) > \lambda\, inf(A)}$~~ $for \quad \lambda > 0 \implies$

$$\implies \frac{inf(\lambda A)}{\lambda} > inf(A) \quad for \quad \lambda > 0$$

$inf(A)$ is the greatest l. b. of A , but we just found an element in $\mathbb{R}$ that is greater than $inf(A)$ $\implies$ $\frac{inf(\lambda A)}{\lambda}$ is <u>not</u> a l. b. of A

🦆 ~~$\boxed{inf(\lambda A) > \lambda\, inf(A)}$~~ $for \quad \lambda > 0$ 🩲 ~~$\boxed{inf(\lambda A) < \lambda\, inf(A)}$~~ $for \quad \lambda > 0$

If neither 🩲 nor 🦆 are true, then:

$$\boxed{inf(\lambda A) = \lambda\, inf(A)} \quad for \quad \lambda > 0$$

$\blacksquare$

$$A \subseteq \mathbb{R} \quad \land \quad \lambda \in \mathbb{R} \ :$$

$$(a) \vdash sup(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda > 0$$

$$(b) \vdash inf(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda > 0$$

$$(c) \vdash inf(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda < 0$$

$$(d) \vdash sup(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda < 0$$

$A \subseteq \mathbb{R} \quad \wedge \quad \lambda \in \mathbb{R}$:

$(c) \vdash inf(\lambda A) = \lambda\, sup(A) \quad if \;\; \lambda < 0$

Notice:

$$3 < 4 \implies (2) \cdot 3 < 4 \cdot (2)$$

$$6 < 8$$

$$3 < 4 \implies (-2) \cdot 3 < 4 \cdot (-2)$$

$$-6 < -8$$

$A \subseteq \mathbb{R} \quad \wedge \quad \lambda \in \mathbb{R}$:

$(c) \vdash inf(\lambda A) = \lambda\, sup(A) \quad if \;\; \lambda < 0$

Notice:

$$3 < 4 \implies (2) \cdot 3 < 4 \cdot (2)$$

$$6 < 8$$

$$3 < 4 \implies (-2) \cdot 3 > 4 \cdot (-2)$$

$$-6 > -8$$

$A \subseteq \mathbb{R} \;\; \wedge \;\; \lambda \in \mathbb{R}$: *Ad absurdum…*

$(c) \vdash inf(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda < 0$. Proof:

<u>Thesis statement</u>:

~~$\boxed{inf(\lambda A) < \lambda\, sup(A)}$~~ $for \quad \lambda < 0 \implies \lambda\, sup(A)$ is <u>not</u> a l. b. of $\lambda A \implies$

$\implies \exists\, x_\lambda := \lambda x \in \lambda A \;\; (x \in A) : \; \cancel{\lambda}\, sup(A) > \cancel{\lambda} x = x_\lambda \implies$

$\implies \exists\, x \in A : \;\; sup(A) < x$ ↯

~~$\boxed{inf(\lambda A) < \lambda\, sup(A)}$~~ $for \quad \lambda < 0$

<u>Thesis statement</u>:

~~$\boxed{inf(\lambda A) > \lambda\, sup(A)}$~~ $for \quad \lambda < 0 \implies \dfrac{inf(\lambda A)}{\lambda} < sup(A) \implies$

$\implies \dfrac{inf(\lambda A)}{\lambda}$ is <u>not</u> an u. b. of $A \implies \exists\, x \in A : \; \dfrac{inf(\lambda A)}{\lambda} < x \implies$

$\implies \exists\, x \in A : \;\; inf(\lambda A) > \lambda x \implies \exists\, x_\lambda := \lambda x \in \lambda A : \;\; inf(\lambda A) > x_\lambda$

↯

~~$\boxed{inf(\lambda A) > \lambda\, sup(A)}$~~ *for* $\lambda < 0$ ~~$\boxed{inf(\lambda A) < \lambda\, sup(A)}$~~ *for* $\lambda < 0$

If neither nor are true, then:

$$\boxed{inf(\lambda A) = \lambda\, sup(A)} \quad \textit{for} \quad \lambda < 0$$

■

$$A \subseteq \mathbb{R} \quad \wedge \quad \lambda \in \mathbb{R} \ :$$

$$(a) \vdash sup(\lambda A) = \lambda\, sup(A) \quad if \quad \lambda > 0$$

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$$(d) \vdash sup(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda < 0$$

$A \subseteq \mathbb{R} \ \wedge \ \lambda \in \mathbb{R}$: *Ad absurdum…*

$(d) \vdash sup(\lambda A) = \lambda\, inf(A) \quad if \quad \lambda < 0$. Proof:

<u>Thesis statement</u>:

$\boxed{\cancel{sup(\lambda A) > \lambda\, inf(A)}} \ \ for \quad \lambda < 0 \implies \lambda\, inf(A)$ is <u>not</u> an u. b. of $\lambda A \implies$

$\implies \exists\, x_\lambda := \lambda x \in \lambda A \ \ (x \in A): \ \cancel{\lambda}\, inf(A) < \cancel{\lambda} x = x_\lambda \implies$

$\implies \exists\, x \in A : \ \ inf(A) > x$ ↯

$\boxed{\cancel{sup(\lambda A) > \lambda\, inf(A)}} \ \ for \quad \lambda < 0$

<u>Thesis statement</u>:

$\boxed{\cancel{sup(\lambda A) < \lambda\, inf(A)}} \ \ for \quad \lambda < 0 \implies \dfrac{sup(\lambda A)}{\lambda} > inf(A) \implies$

$\implies \dfrac{sup(\lambda A)}{\lambda}$ is <u>not</u> a l. b. of $A \implies \exists\, x \in A : \ \dfrac{sup(\lambda A)}{\lambda} > x \implies$

$\implies \exists\, \lambda x \in \lambda A : \ \ sup(\lambda A) < \lambda x$ ↯

~~$\boxed{sup(\lambda A) < \lambda\, inf(A)}$~~ $for \quad \lambda < 0$

~~$\boxed{sup(\lambda A) > \lambda\, inf(A)}$~~ $for \quad \lambda < 0$

If neither nor are true, then:

$$\boxed{sup(\lambda A) = \lambda\, inf(A)} \quad for \quad \lambda < 0$$

■

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