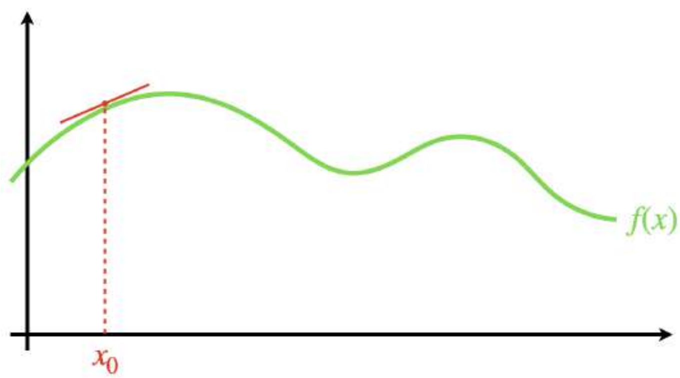
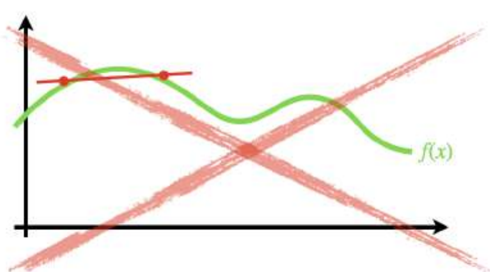
Intuition Behind Derivative & de L'Hôpital Rule

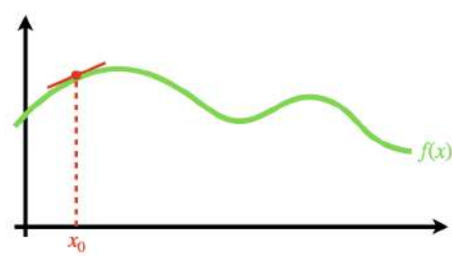
by Dibeos

Imagine the graph of a function $f(x)$. At any point x_0 in the x-axis there is a unique straight line that just touches the function at one point.



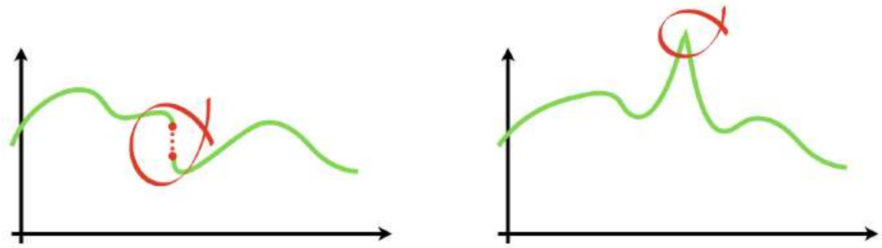
This line is special because it does not cross the function at 2 points. Just one!



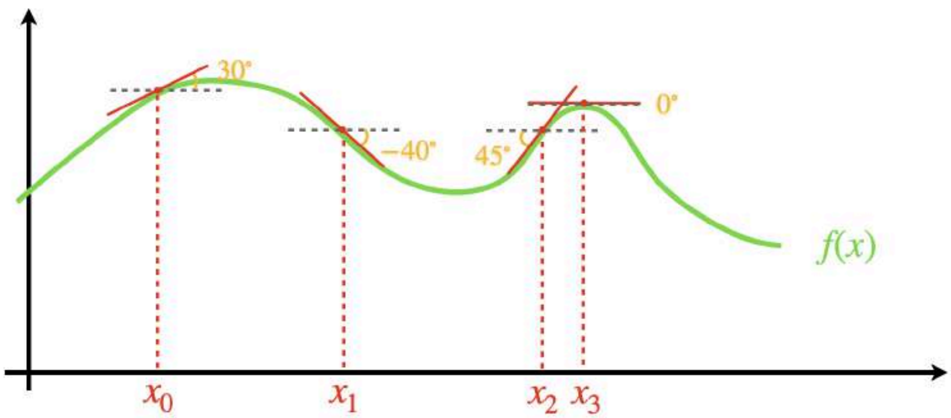


The fact that this is true makes this line unique. It's called the *tangent line* at a point.

As long as f is *continuous* and has *no "sharp" corners*

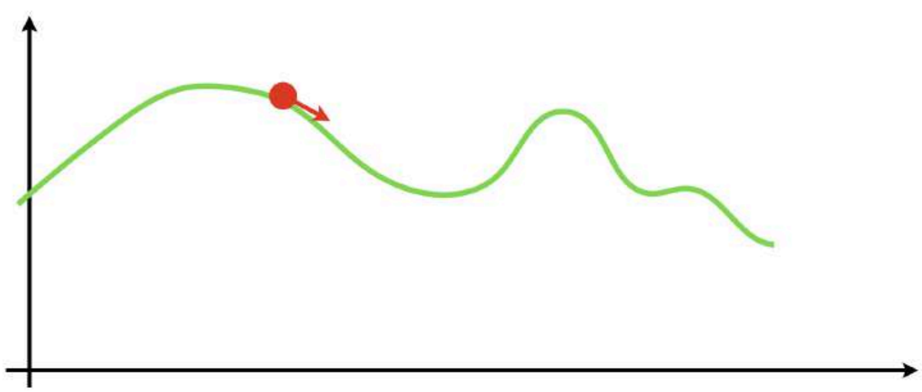


we can find a <u>unique</u> tangent line for each point of the function. Each of these tangent lines form a specific angle with the horizontal line at that point.



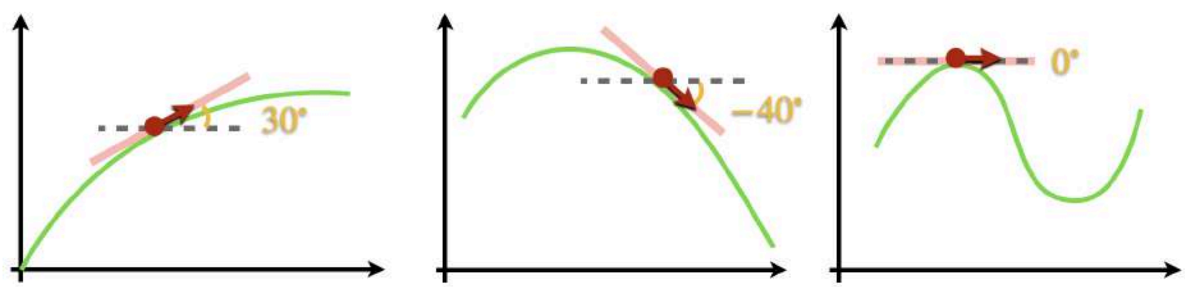
These angles are useful to track because they give us a way of *quantifying* (or *measuring*) the steepness of the curve at each point.

If you imagine that this function is actually the path of a tiny particle moving on a plane,

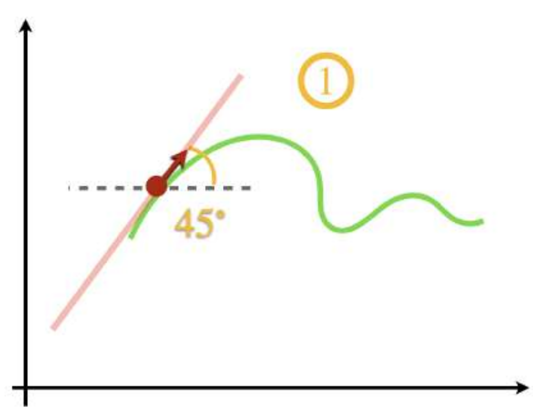


we can assign a little arrow at each point, tangent to the curve at each point, representing what a physicist would call *linear* or *tangent velocity*.

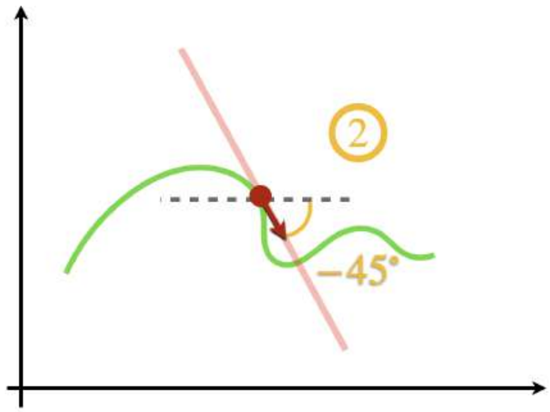
This intuitive picture is useful because it gives us a sense of what the tangent line at each point tells us, i.e. how “fast” the function is going up, going down, or simply being stationary at a fixed “height”.



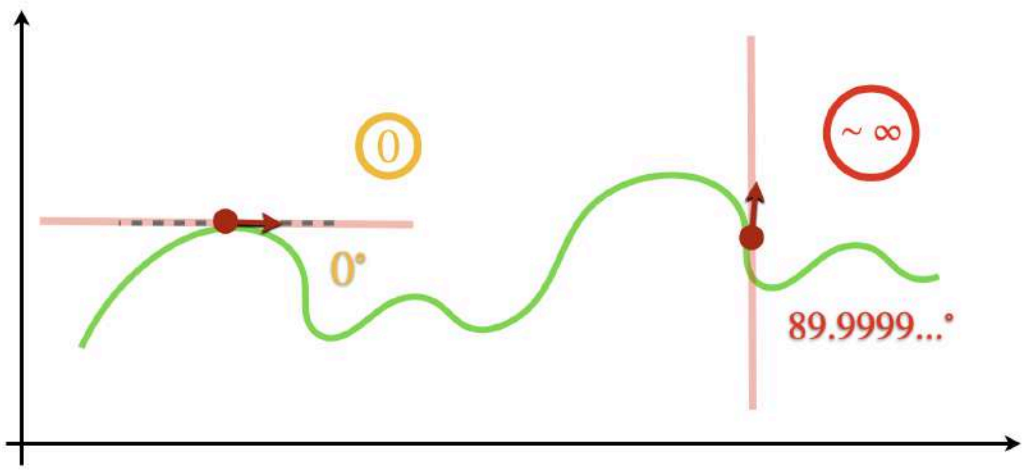
We can consider a tangent line at a specific point with an angle of 45° as having *inclination* (or *steepness*) 1.



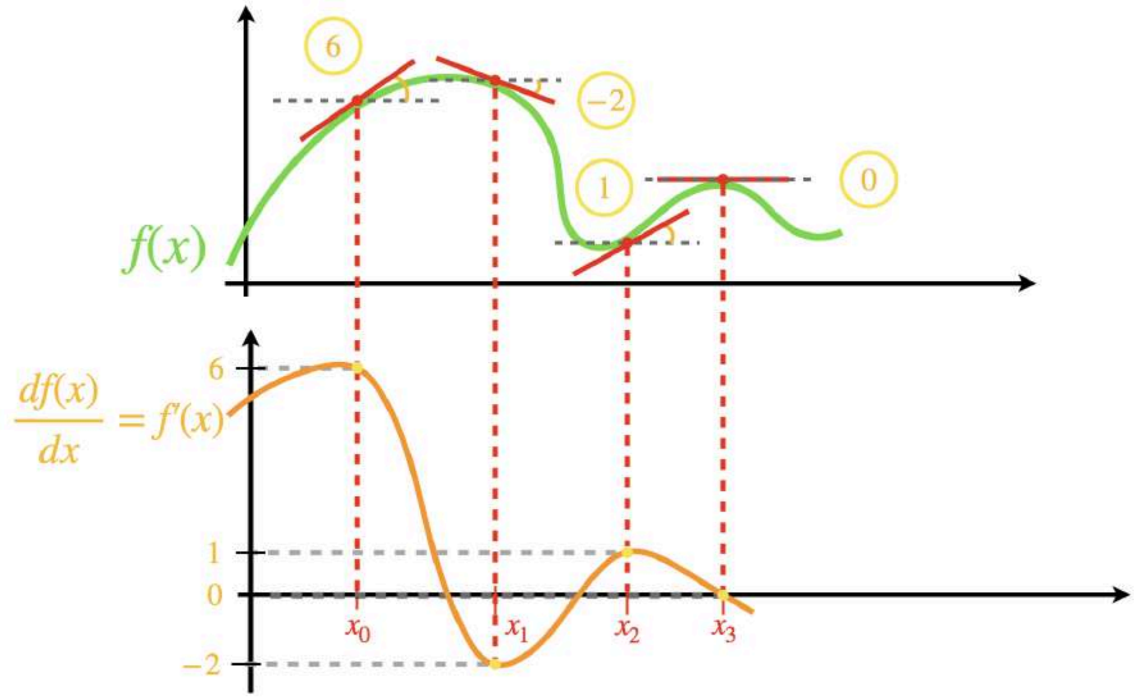
Following the same logic, a tangent line that forms an angle of 45°, but downwards, has inclination -1.



For a 0° angle, the inclination is 0. And for $\approx 90^{\circ}$ the inclination is a very large number.



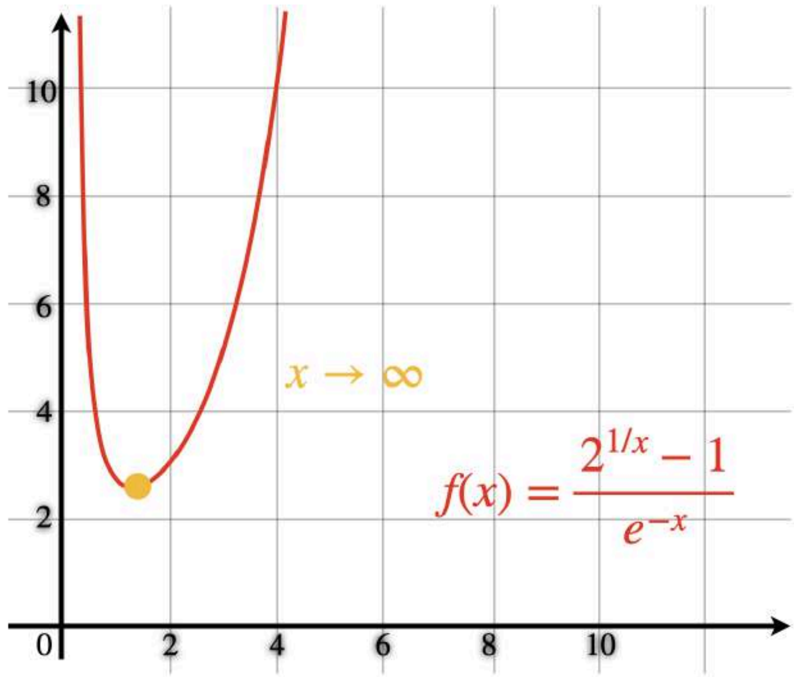
We need a way of tracking down the inclination of all these tangent lines at each point.



We can do this for all points. And that's the concept of **derivative**!

So, the derivative gives us a way of measuring how fast the function is changing its height. This is extremely useful!

For example, let's suppose we want to calculate the limit, with x that tends to ∞ $(x \to \infty)$, of the function $f(x) = \frac{2^{1/x}-1}{e^{-x}}$.



If we simply replace x with ∞ we will get $\frac{0}{0}$. This is an *indeterminate form*. It does not mean that the limit does not exist for $x \to \infty$, but we do need to find an alternative way of evaluating this limit.

$$f(\infty) = \frac{2^{1/\infty} - 1}{e^{-\infty}} = \frac{2^0 - 1}{\frac{1}{e^{\infty}}} = \frac{1 - 1}{\frac{1}{\infty}} = \frac{0}{0}$$

indeterminate form

The question here is: which one of these functions goes faster to zero, as $x \to \infty$? The numerator $\left(2^{1/x} - 1\right)$ or the denominator $\left(e^{-x}\right)$? This is a very important question. For example, notice the pattern in the following sequence:

$$\frac{0.25}{0.25} \rightarrow \frac{0.125}{0.0625} \rightarrow \frac{0.0625}{0.00390625} \rightarrow \frac{0.03125}{0.000015258...} \rightarrow \ ... \rightarrow \frac{N}{D} \sim \frac{N}{0}$$

You can check by yourself, but the numerator (N) decreases by half and the denominator (D) is squared at each step. The general formula for the n-th element $\left(\frac{a_n}{b_n}\right)$ would be

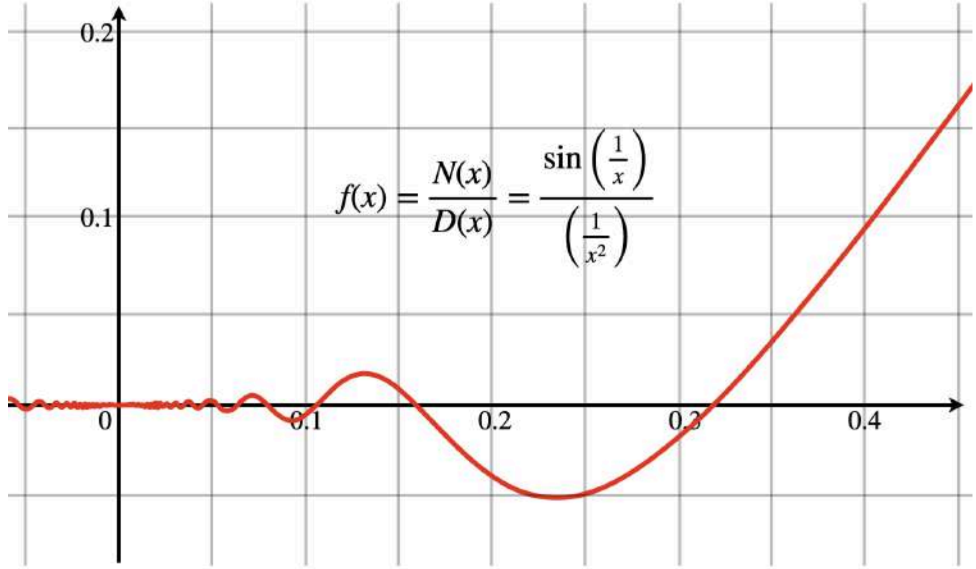
$$\frac{a_n}{b_n} = \frac{\left(\frac{a_{n-1}}{2}\right)}{\left(b_n\right)^2}$$

Both the numerator and denominator approach zero, but we can clearly see that the denominator *tends to zero faster* than the numerator. So, the indeterminate form $\frac{0}{0}$ can be transformed into a determinate form once we notice that after enough iterations we will get a number that is

$$\approx \frac{a_n}{0} = \infty$$

In other words, the sequence tends to a number different from zero (the numerator) over a number veeeeeery close to zero (the denominator), and thus we say that the result is ∞, for the same reason that *"dividing a number infinite times results in zero* $\left(loosely\ speaking: \ \frac{fixed\ number}{\infty} = 0\right)$*".*

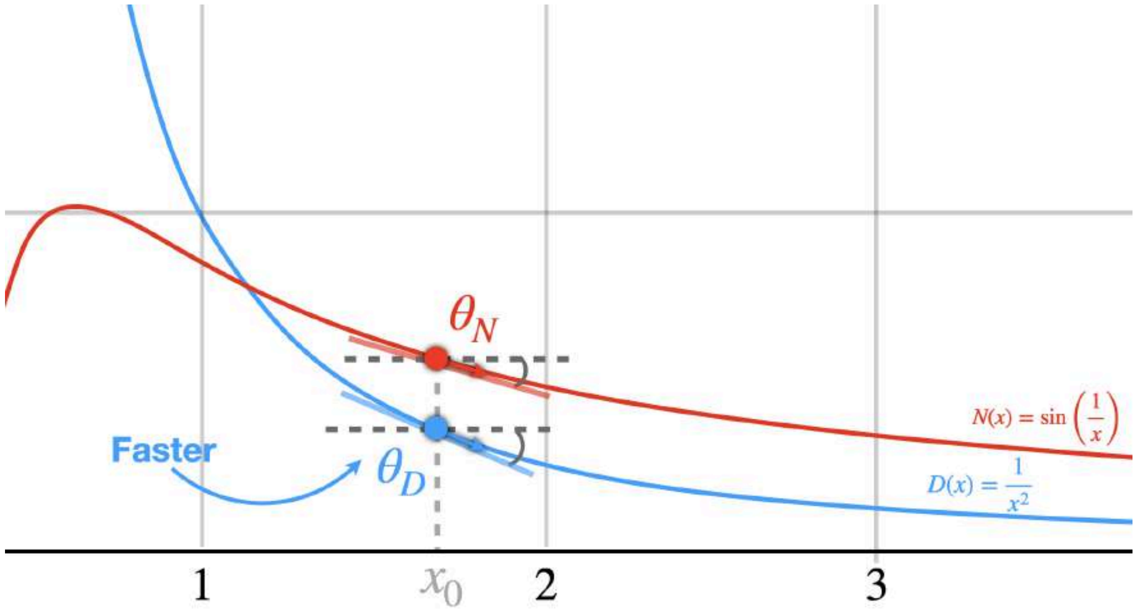
This is obviously a discrete sequence, but we could easily imagine another problem involving a continuous function, like the following function:



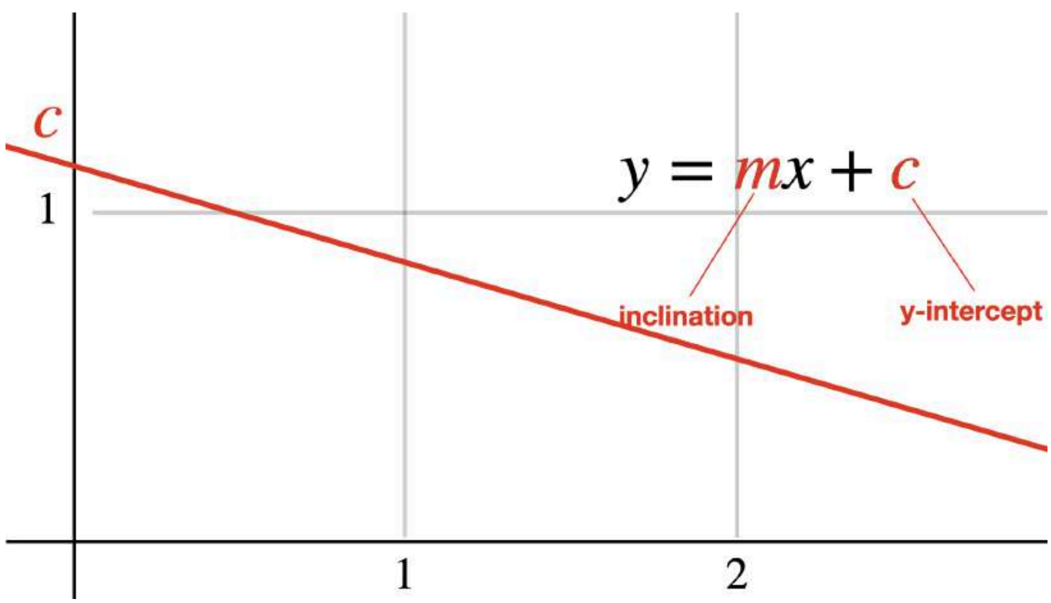
In order to properly evaluate the limit of this function, for $x \to \infty$, we can apply a method called *de L'Hôpital rule* (where $\frac{d}{dx}$ (...) is the derivative of (...) with respect to x):

$$\lim_{x\to\infty} \frac{N(x)}{D(x)} = \lim_{x\to\infty} \frac{sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(sin\left(\frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cdot cos\left(\frac{1}{x}\right)}{\left(-\frac{2}{x^3}\right)} =$$

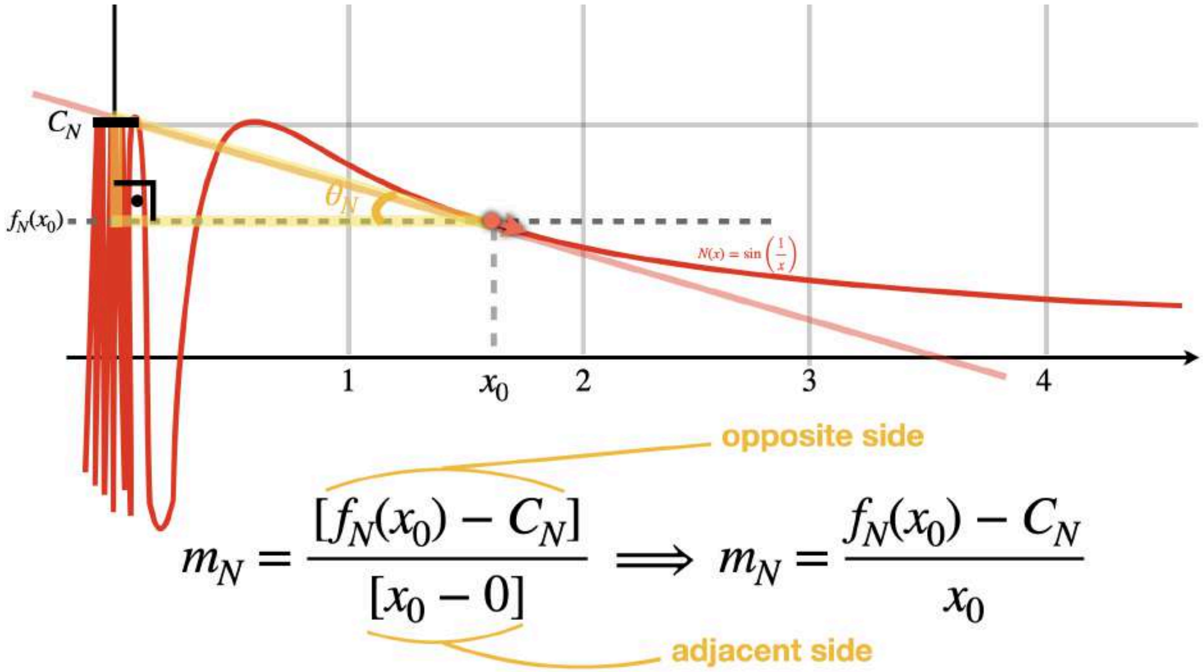
$$= \lim_{x\to\infty} \frac{x^3}{2x^2} cos\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{x}{2} cos\left(\frac{1}{x}\right) = "\frac{\infty}{2} \cdot 1" = \infty$$



The equation of a line is defined by 2 parameters, usually denoted as m and c, representing the *inclination* of the line and the point in which the line *crosses the y-axis*, respectively.



Let's look closely at the numerator's tangent line and try to determine its inclination.



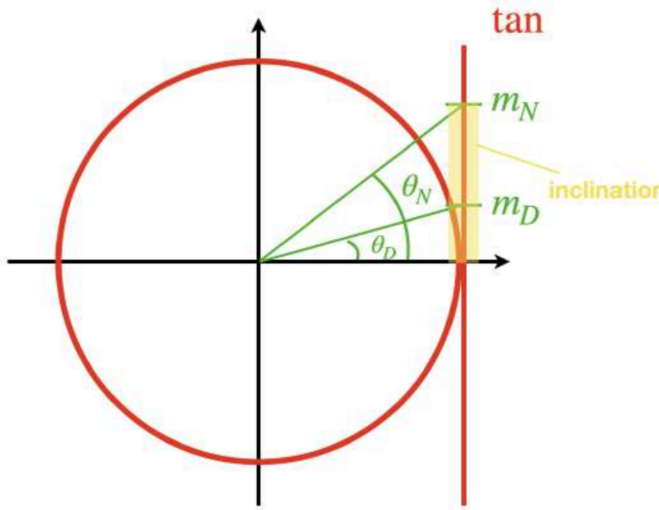
As you can see, there is a right triangle here, which lets us conclude that the inclination m_N is the opposite side of the triangle with respect to θ_N divided by the adjacent side (i.e. the side that “touches” the angle). Why is it relevant? Well, in mathematics, there is a name for the fraction created by dividing the opposite side by the adjacent side, and this name is: tangent of the angle. It is a trigonometric function:

$$m_N = tan(\theta_N)$$

Similarly, with the denominator, we get:

$$m_D = tan(\theta_D)$$

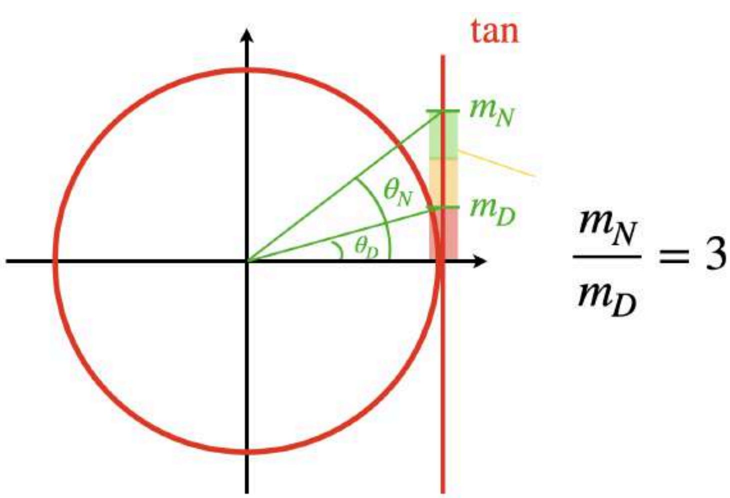
This trigonometric function has a very nice visual representation.



The tangent of an angle is the length of the segment line formed by extending the green line from the center until it crosses the red vertical line (figure above). This *length* is the inclination of our original function at a specific point.

Since our goal is to compare the function in the numerator $N(x)$ with the function in the denominator $D(x)$ in order to decide which one goes "faster" to zero, as $x \to \infty$, all we actually need to do is compare their inclinations at each point, and that's why *de L'Hôpital* tells us to take their derivatives.

Say that $\frac{m_N}{m_D} = 3$, then it means that, in the red vertical tangent line,



you can fit in 3 lengths of m_D inside of m_N . So, at the end of the day, it all comes down to find out the ratios $\frac{m_N}{m_D}$ of all the tangent lines at each point x , as $x \to \infty$.

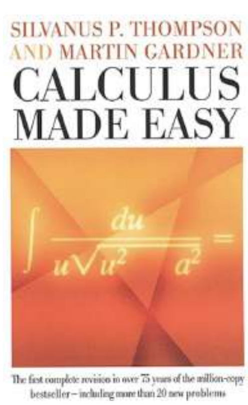
Since the inclination m is the derivative at each point, the *de L'Hôpital rule* holds for the previous example that we've seen:

$$\lim_{x\to\infty} \frac{N(x)}{D(x)} = \lim_{x\to\infty} \frac{sin\left(\frac{1}{x}\right)}{\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{\frac{d}{dx}\left(sin\left(\frac{1}{x}\right)\right)}{\frac{d}{dx}\left(\frac{1}{x^2}\right)} = \lim_{x\to\infty} \frac{-\frac{1}{x^2}\cdot cos\left(\frac{1}{x}\right)}{\left(-\frac{2}{x^3}\right)} =$$

$$= \lim_{x\to\infty} \frac{x^3}{2x^2} cos\left(\frac{1}{x}\right) = \lim_{x\to\infty} \frac{x}{2} cos\left(\frac{1}{x}\right) = "\frac{\infty}{2}\cdot 1" = \infty$$

Now that we have a strong intuition behind the *de L'Hôpital rule*, and that we've seen a concrete example of its application, let's rigorously prove it. We will have to digress a bit, but it will be completely worth the effort.

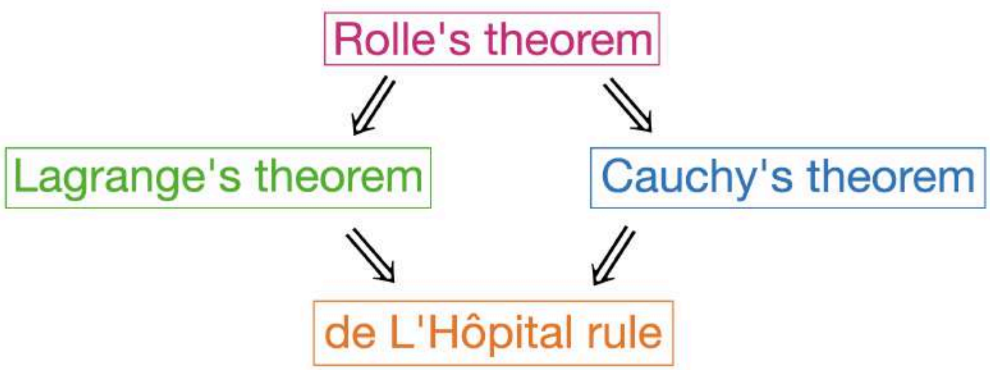
Ah! And btw, if you want to learn more about these concepts, check out the following amazing book:



[Calculus Made Easy]

Check out our website for more! https://dibeos.wordpress.com/
You can also submit to us your own work so that we can publish on the website and others can peer review it, comment on it and help you to improve!

In order to prove the *de L'Hôpital rule* we will take some steps back and study other 3 theorems in Analysis. These will build the foundation to convince us that the *de L'Hôpital rule* holds, given the correct conditions.



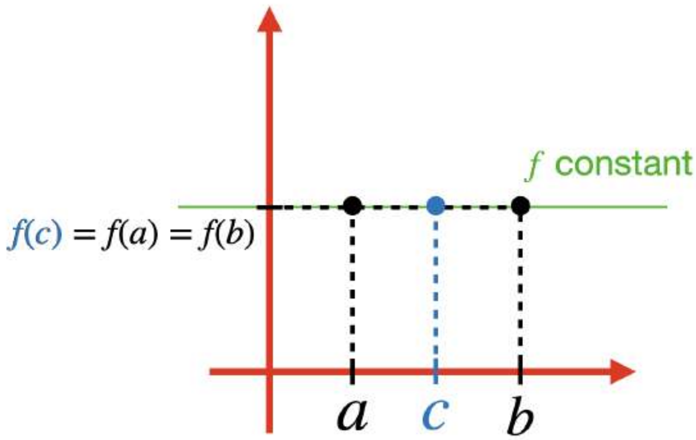
(I) Rolle's theorem

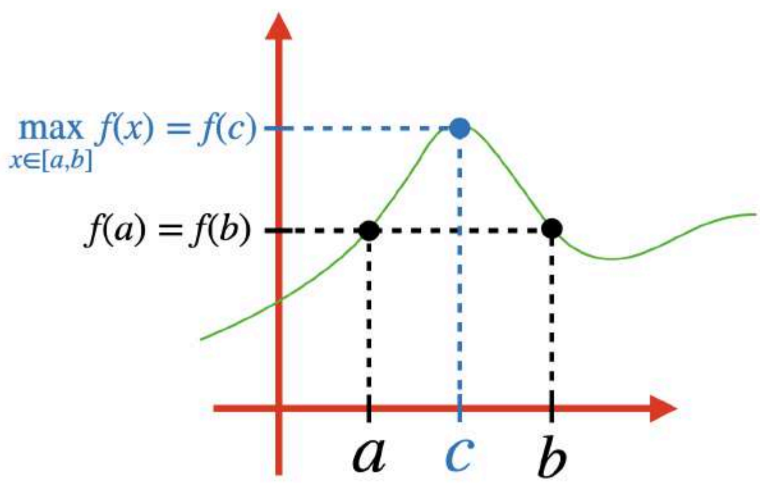
Let $f:[a,b]\to\mathbb{R}$ be a continuous function, and differentiable on (a,b), such that:

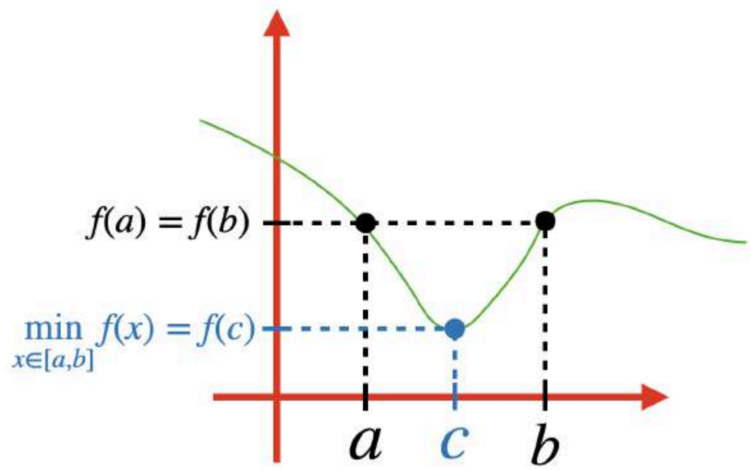
$$f(a) = f(b) \quad \Rightarrow \quad \exists\, c \in (a,b) \subset \mathbb{R}: \quad f'(c) = 0\ .$$

Proof:

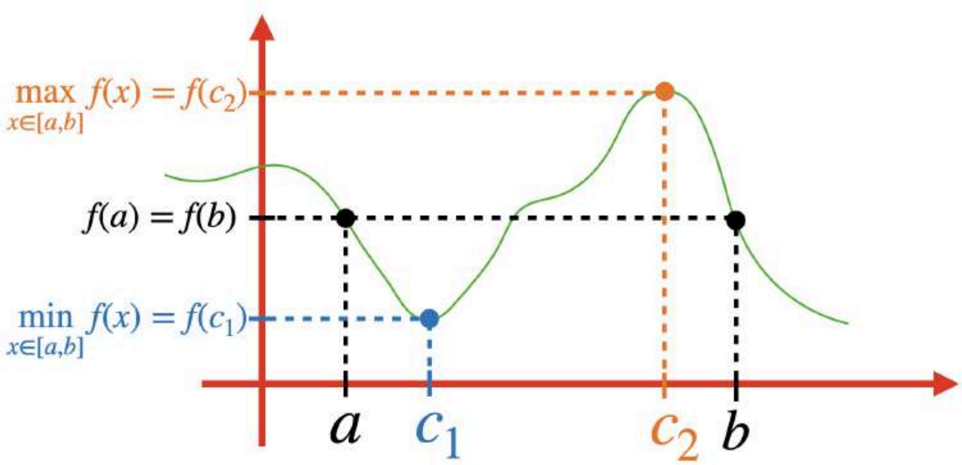
$f: [a, b] \to \mathbb{R}$ is a continuous function $\Rightarrow \exists \max\limits_{x \in [a,b]} f(x)$ and $\exists \min\limits_{x \in [a,b]} f(x)$. There are only the following possibilities:



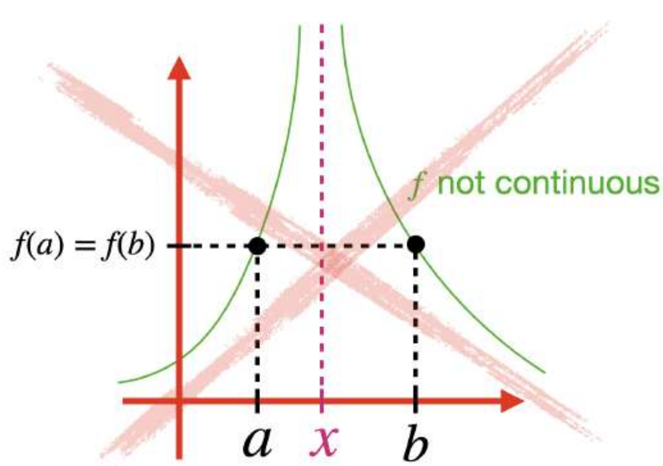




$f(a) = f(b)$
$\min_{x\in[a,b]} f(x) = f(c)$
a
c
b



$\max_{x\in[a,b]} f(x) = f(c_2)$
$f(a) = f(b)$
$\min_{x\in[a,b]} f(x) = f(c_1)$
a
c_1
c_2
b



f not continuous
$f(a) = f(b)$
a
x
b

Case 1: (f constant)

$$f(x) = f(a) = f(b) \ , \ x \in [a,b] \ \Rightarrow \ f'(x) := \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{f(x)-f(x)}{h} = 0$$

$\therefore \exists \ c \in (a,b)$ (indeed, $\forall \ c \in (a,b)$) such that $f'(c) = 0$.

Case 2: (f ¬ constant)

WLOG *(without loss of generality)*, we can assume that $\exists \ c \in (a,b)$ such that

$$\max_{x \in [a,b]} f(x) := f(c) \quad \Rightarrow \quad \boxed{f(c) \geq f(x) \ , \ \forall \ x \in (a,b)}$$

Subcase 1: $(h \to 0^+)$

$f(c) = \max_{x \in [c \, , \, c+h]} f(x)$ (i.e. $f(c)$ is also a local maximum point) $\Rightarrow$

$$\Rightarrow \quad f(c+h) \leq f(c) \ , \ \forall \ h \approx 0^+ \quad \Rightarrow \quad f(c+h) - f(c) \leq 0 \ , \ h \approx 0^+$$

$\Rightarrow \quad \frac{f(c+h)-f(c)}{h} \leq 0 \ , \ h \approx 0^+$ (the sign of this inequality remains the same because we divided by a positive number ($h > 0$) in the previous step)

$$\Rightarrow \quad f'_+(c) := \lim_{h \to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0 \quad \Rightarrow \quad \boxed{f'_+(c) \leq 0} \quad \text{(I)}$$

Subcase 2: $\left(h \to 0^{-}\right)$

$f(c) = \max_{x \in [c+h\,,\,c]} f(x) \quad \left(h \approx 0^{-}\right)$ (i.e. $f(c)$ is also a local maximum point) $\Rightarrow$

$$\Rightarrow \quad f(c + h) \le f(c) \quad , \quad \forall\, h \approx 0^{-} \quad \Rightarrow \quad f(c + h) - f(c) \le 0 \quad , \quad h \approx 0^{-}$$

$\Rightarrow \quad \frac{f(c+h)-f(c)}{h} \ge 0 \quad , \quad h \approx 0^{-}$ (the sign of this inequality is inverted because we divided by a negative number ($h < 0$) in the previous step)

$$\Rightarrow \quad f'\_(c) := \lim_{h \to 0^{-}} \frac{f(c+h)-f(c)}{h} \ge 0 \quad \Rightarrow \quad \boxed{f'\_(c) \ge 0} \quad \text{(II)}$$

In conclusion, (I) and (II) $\Rightarrow \boxed{f'(c) := \lim_{h \to 0} \frac{f(c+h)-f(c)}{h} = 0}$ ■

(II) Lagrange's theorem *(or the Mean Value theorem)*

Let $f: [a, b] \to \mathbb{R}$ be a continuous function, and differentiable on (a, b), then $\exists\, c \in (a, b)$, such that:

$$f'(c) = \frac{f(b)-f(a)}{b-a}$$

Proof:

In order to prove this result, we will utilize the following auxiliary function:

$$g(x) := f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x - a)$$

Now, we will check that this function satisfies all the hypotheses of Rolle's theorem, and thus we can use it in the current proof.

(1) *Continuity:* $f: [a, b] \to \mathbb{R}$ *is continuous* $\Rightarrow$ g *is continuous ;*

(2) Differentiability: f *is differentiable on* (a, b) $\Rightarrow$ g *is differentiable ;*

(3) We want to show that $g(a) = g(b)$:

✳ $g(a) = f(a) - \left(\frac{f(b)-f(a)}{b-a}\right)(a - a) = f(a)$

✳ $g(b) = f(b) - \left(\frac{f(b)-f(a)}{b-a}\right)(b - a) = f(b) - f(b) + f(a) = f(a)$

$$\therefore \boxed{g(a) = g(b)}$$

With these 3 things in place, we can use Rolle's theorem:

$$\exists\, c \in (a, b): \quad g'(c) = 0 \quad \Leftrightarrow \quad \frac{d}{dx}\left(f(x) - \left(\frac{f(b)-f(a)}{b-a}\right)(x - a)\right)_{x=c} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left(f'(x) - \frac{f(b)-f(a)}{b-a}\right)_{x=c} = 0 \quad \Leftrightarrow \quad f'(c) - \frac{f(b)-f(a)}{b-a} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \boxed{f'(c) = \frac{f(b)-f(a)}{b-a}} \qquad \blacksquare$$

(III) Cauchy's theorem

If $f, g : [a, b] \to \mathbb{R}$ are continuous functions, and differentiable on (a, b) , with $g'(x) \neq 0$, $\forall\, x \in (a, b)$, then $\exists\, c \in (a, b)$:

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}$$

Proof:

Once again we will need an auxiliary function:

$$h(x) := f(x) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(x)$$

This function satisfies Rolle's theorem's hypothesis:

(1) *Continuity:* f *and* g *are continuous* $\Rightarrow$ h *is continuous ;*

(2) *Differentiability:* f *and* g *are differentiable on* (a, b) $\Rightarrow$ g *is differentiable ;*

(3) We want to show that $h(a) = h(b)$:

$$\ast\ \ h(a) = f(a) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(a) = f(a) + \frac{-f(b)g(a)+f(a)g(a)}{g(b)-g(a)} =$$

$$= \frac{f(a)g(b)-f(a)g(a)-f(b)g(a)+f(a)g(a)}{g(b)-g(a)} =$$

$$= \frac{f(a)g(b)-f(b)g(a)}{g(b)-g(a)}$$

$$\ast\ \ h(b) = f(b) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(b) =$$

$$= f(b) + \frac{-f(b)\, g(b) + f(a)\, g(b)}{g(b) - g(a)} =$$

$$= \frac{f(b)\, g(b) - f(b)\, g(a) - f(b)\, g(b) + f(a)\, g(b)}{g(b) - g(a)} =$$

$$= \frac{f(a)\, g(b) - f(b)\, g(a)}{g(b) - g(a)}$$

$$\therefore \boxed{h(a) = h(b)}$$

With these 3 things in place, we can use Rolle's theorem:

$$\exists\, c \in (a, b): \quad h'(c) = 0 \quad \Leftrightarrow \quad \frac{d}{dx}\left(f(x) - \left(\frac{f(b)-f(a)}{g(b)-g(a)}\right) g(x)\right)_{x=c} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \left(f'(x) - \frac{f(b)-f(a)}{g(b)-g(a)}\, g'(x)\right)_{x=c} = 0 \quad \Leftrightarrow$$

$$\Leftrightarrow \quad \boxed{\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}} \qquad \blacksquare$$

Finally, we can prove:

(IV) de L'Hôpital Rule:

Let f, g be differentiable functions on (a, b) . If $f(c) = g(c) = 0$, for some $c \in (a, b)$, and $g'(x) \neq 0$, $\forall\, x \in (a, b)$, with $x \neq c$, and $\exists \lim_{x \to c} \frac{f'(x)}{g'(x)} = L$, ***then***

$$\lim_{x \to c} \frac{f(x)}{g(x)} = L$$

In other words, if finding the limit of $\frac{f(x)}{g(x)}$ *, for* $x \to c$ *, is too hard, you can instead calculate the limit, for* $x \to c$ *, of the derivative of the numerator* $(f'(x))$ *divided by the derivative of the denominator* $(g'(x))$*.*

Proof:

There are 2 possible cases: $\left(x_0 \in \mathbb{R}\right)$

<u>Case 1</u>:

$$\lim_{x \to x_0} f(x) = \lim_{x \to x_0} g(x) = 0 \quad \left(\frac{0}{0}\right)$$

Since f and g are continuous functions, we have that: $f\left(x_0\right) = g\left(x_0\right) = 0$.

Using Cauchy's theorem we get that

$\exists\, c \in (a, b)$: $\quad (g'(x) \neq 0, \forall\, x \in (a, b))$

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

In our context here $\left(i.e.\ c(x) \in \left(x, x_0\right) \right)$:

$$\frac{f(x) - f\left(x_0\right)}{g(x) - g\left(x_0\right)} = \frac{f'(c(x))}{g'(c(x))} \quad \Rightarrow \quad \frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))} \quad \Rightarrow$$

$$\Rightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(c(x))}{g'(c(x))} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)} \qquad \blacksquare$$

Case 2:

$$\lim_{x \to x_0} f(x) = \pm \infty = \lim_{x \to x_0} g(x) \quad \left(\frac{\pm\infty}{\pm\infty}\right)$$

Using Cauchy's theorem we get that

$\exists\, c \in (a, b):$ $\qquad (g'(x) \neq 0,\ \forall\, x \in (a, b))$

$$\frac{f(x)}{g(x)} = \frac{f'(c(x))}{g'(c(x))} \quad \Rightarrow \quad \lim_{x \to x_0} \frac{f(x)}{g(x)} = \lim_{x \to x_0} \frac{f'(x)}{g'(x)}$$

■

Please, if you find this document useful, let us know. Or if you found typos and things to improve, let us know as well. Your feedback is very important to us. We're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com