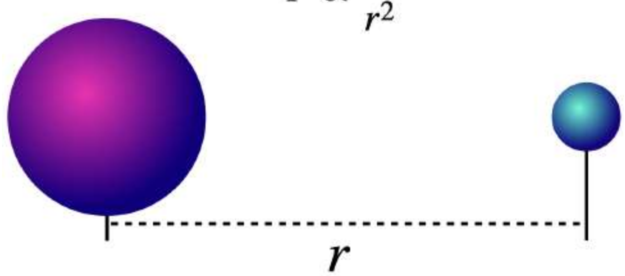
Intuition Behind Curved Spaces In Math & Physics

by DiBeos

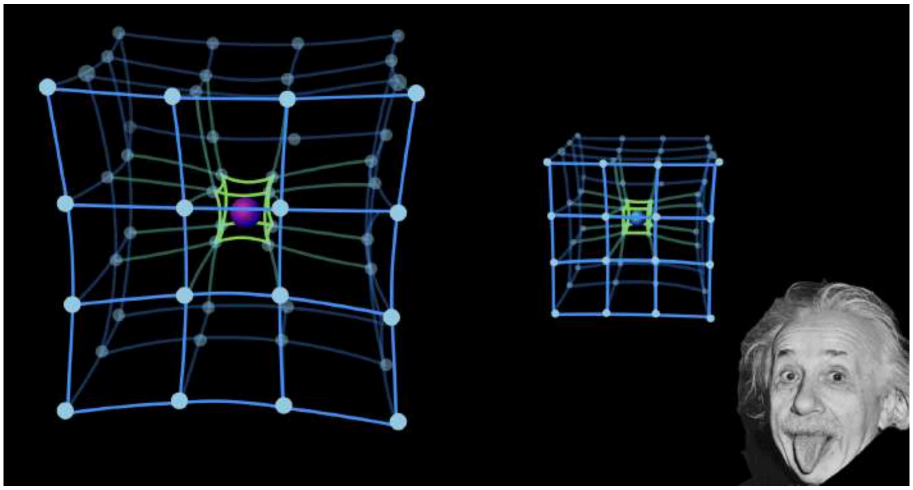
Newton was the one who found out that everything attracts everything else with a force inversely proportional to the square of the distance.

$$F = G\frac{m_1 m_2}{r^2}$$

$$F \propto \frac{1}{r^2}$$



Einstein had a different interpretation of gravity. Space and time are not as rigid as Newton thought — they are distorted near massive objects. And the curvature can be perceived more clearly when dealing with heavy masses. It is this curvature of spacetime that gives the effect that we call gravity. Mathematically, the scenario changed as well. Gravity is not a force anymore, but merely the distortion of spacetime around the massive object.



curvature

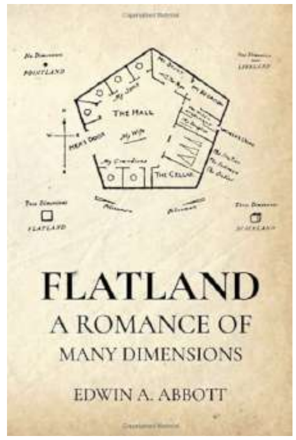
effect of gravity

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

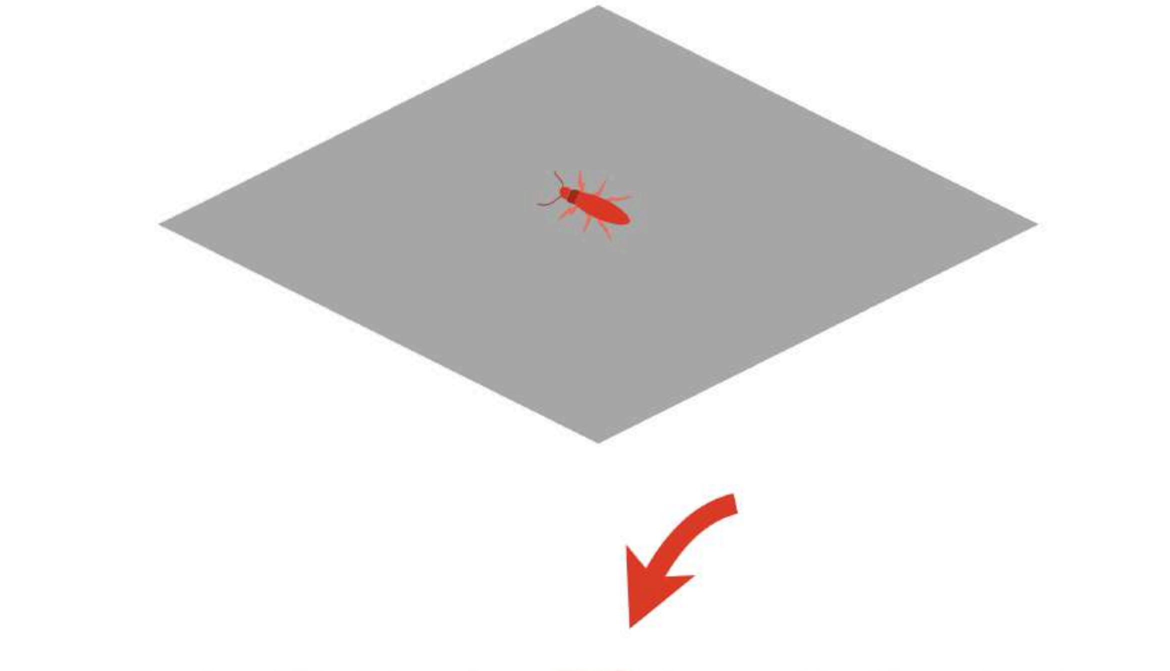
spacetime

There are 3 things involved here. One is the *effect of gravity* (the **stress-energy tensor**). Another is the idea of *spacetime* (the **metric tensor**). The third involves the idea of *curvature* (the **Ricci curvature tensor**). We will simplify things here by not talking about gravity and by leaving out the time — so, we will focus on curvature. Now, even that much turns out to be very hard in 3 dimensions. So, we will first reduce the problem to a “curved space” in 2 dimensions.

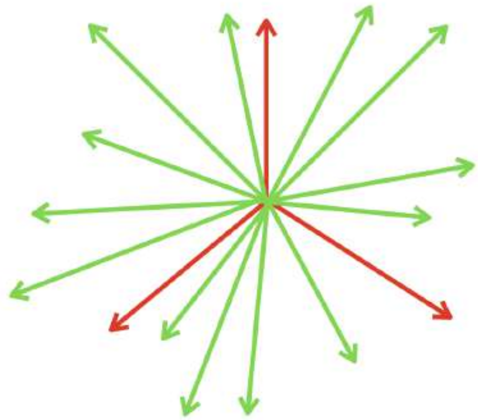
In order to understand this idea of curved space in 2 dimensions you need to put yourself in the shoes of a creature that lives in such a space. One of my favorite books, that helped me to appreciate dimensions in math and physics, and also improved my spatial intuition when navigating high dimensions, is [“FLATLAND – a romance in many dimensions” by Edwin Abbott](#).



Let me give you a taste of it. Imagine a very weird bug with no eyes who lives on a plane. He can move only on the plane, and he has no way of knowing that there is a whole world “outside” of the plane. This bug is 2-dimensional. He has no thickness, no height. If you take the sheet of paper where it lives and look at it from the side, you will see the bug as just a straight line.

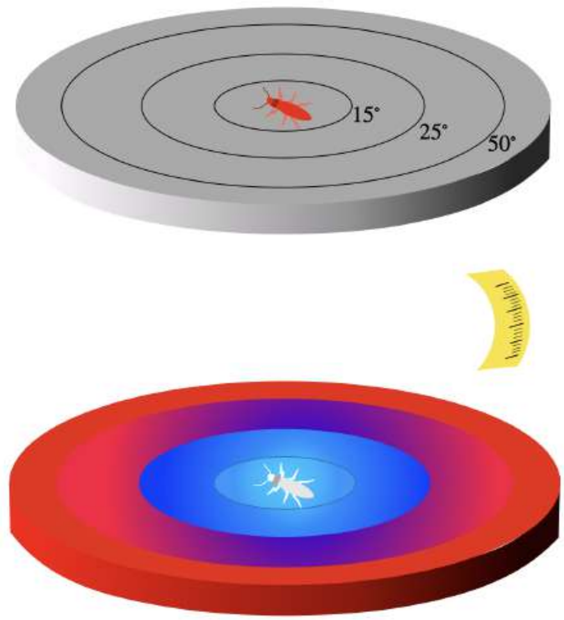


You may find yourself having a hard time trying to be empathetic to this being. I mean, what a stupid bug! Why can't it just literally look up and see that his world is actually just a section of a much larger 3-dimensional world? The world where we observe him from. From above. Or at least he could imagine it, right?! Well, actually we live in a 3-dimensional world, and we don't have any imagination about going off our 3D world in a new direction. I mean, nothing guarantees us that there are no other dimensions out there, other directions (beyond the 3 dimensions of space and 1 of time). As far as we know there are only 4. At least, that's what physics tells us. But in the mathematical world, 10, 100, or even ∞ dimensions are as real as the 3 dimensions we are used to.



Another example of a bug living in 2 dimensions is one who lives on a sphere. He can move around on the surface of the sphere, but, just as before, he cannot look “up” or “down” or “out”. Also, because “up”, “down” and “out” do not exist in his 2-dimensional world.

Now we consider a third kind of bug. This is also a bug that lives in a plane, like the first one, but this time with a little *caveat*. This plane is a disk, with different temperatures at different places (i.e. with a non-constant scalar field defined on its domain). It is cold in the center and heats up gradually as one moves away from it. Also, the bug and any objects in it, including the rulers he uses now and then to measure distances, are composed of the same material that expands when heated up and contracts when cooled down. Everything is longer in the hot places than it is in the cold places, and everything has the same coefficient of expansion.



Area Expansion: $$A = A_0(1 + 2\alpha\Delta T)$$

A: The new area of the object at temperature T.

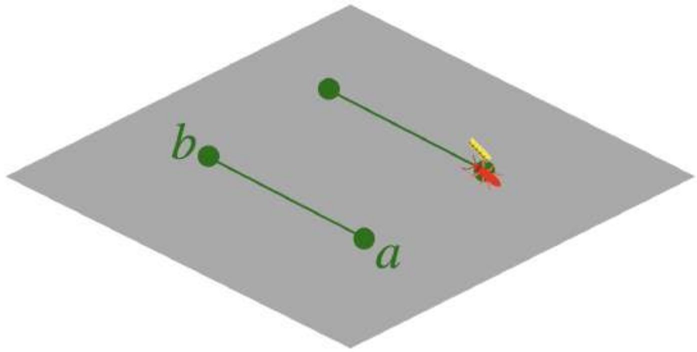
A_0 : The original area of the object at a reference temperature.

α : The coefficient of linear expansion (material-specific constant).

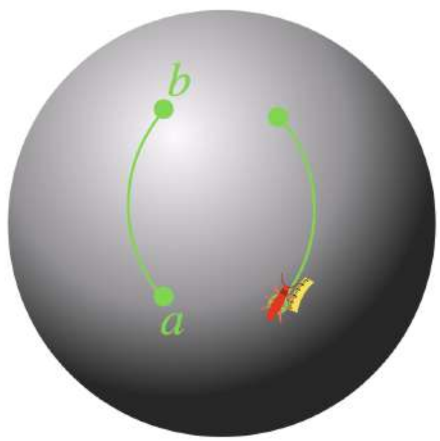
ΔT : The temperature difference.

But we forgot to tell you that these are no ordinary bugs, they are actually very smart, and they want to study geometry. They draw lines, they make rulers, and measure lengths. One of the simplest ideas they learned is how to draw straight lines in their respective spaces. A straight line is defined as the shortest line between two points.

The first bug makes beautiful straight lines.

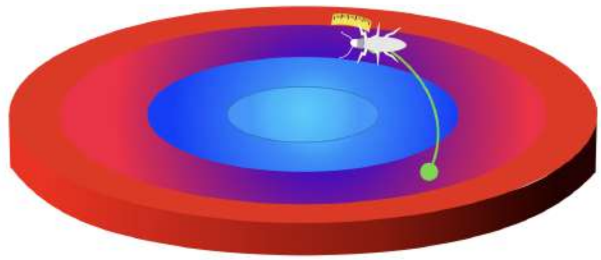


But what about the bug on the sphere? He draws his “straight line” as the shortest distance, for him, between two points.

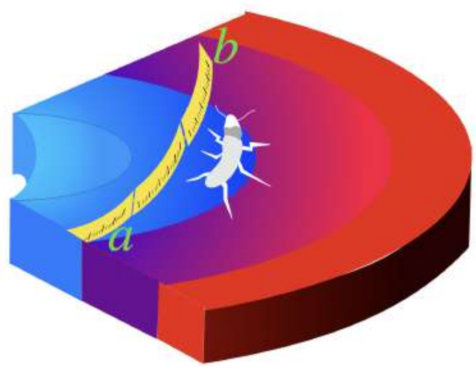


It looks like a curve to us, because we have the privilege of the third dimension (which is actually an arc of a great circle), but for him 2 is the maximal number of dimensions that exist. All he actually knows is that if he tries any other path in his world it is always longer than his straight line. It’s just his empirical knowledge.

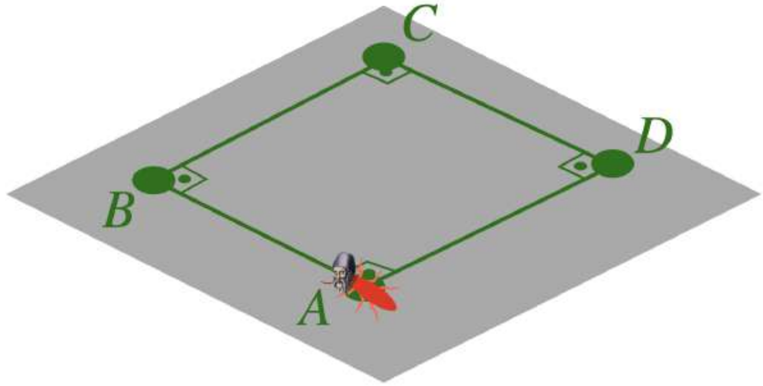
The third bug will also draw “straight lines” that look curved to us.



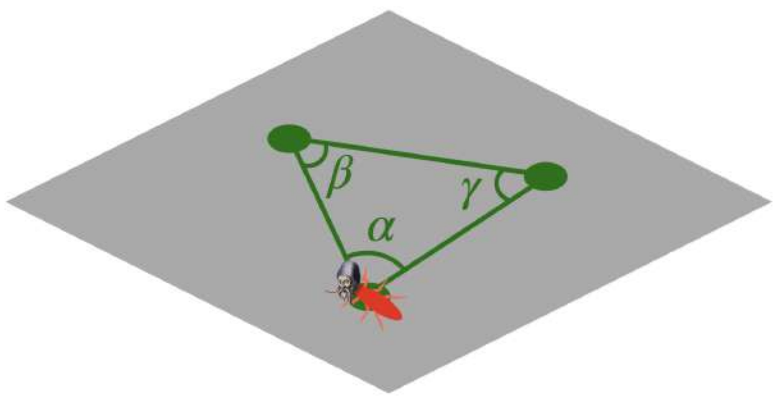
The shortest distance between the points A and B is a curve. How come? Because when his line curves out toward the warmer parts of his plate, the rulers get longer, they expand (only from our point of view, of course) and thus it takes fewer “sticks” laid end-to-end to get from A to B. For the bug, though the line is perfectly straight.

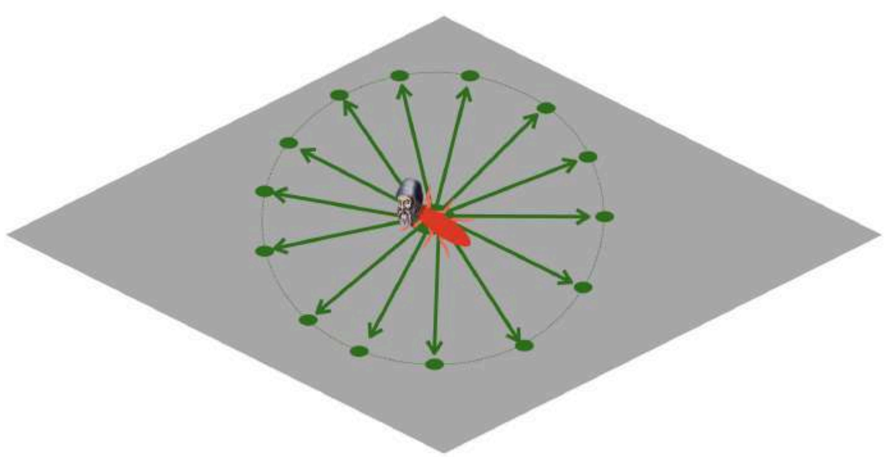


Now, let’s see what the rest of their geometries look like. Let’s assume that all the bugs learned how to make two lines intersect at right angles (90°). Then, the first bug, in the normal plane, finds an interesting fact. If he starts at the point A and makes a line 100 inches long, then makes a right angle and marks off another 100 inches, then makes another right angle and goes another 100 inches, then makes a third right angle and a fourth line 100 inches long, he ends up right at the starting point. He calls it a *postulate in geometry*. More specifically, a postulate in HIS geometry. Let’s call this bug Euclidicus.



Then, Euclidicus discovers another interesting fact. If he makes a triangle, which is just three straight lines connected at the endpoints, the sum of the angles is equal to 180°. Right after that, Euclidicus makes a circle. He started at a fixed point and rushed off on straight lines in many directions. Then he laid out a lot of dots that are all the same distance from that original point.

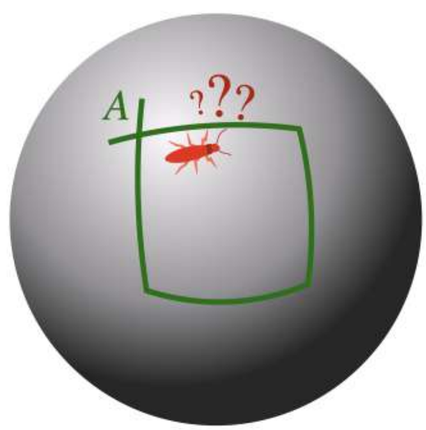




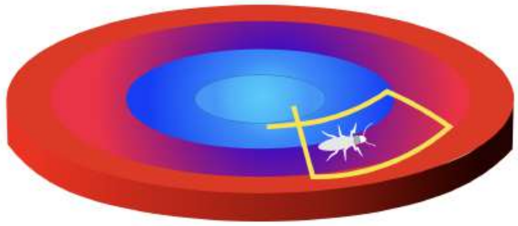
One day, Euclidicus decides to measure the distance around many of the circles he drew, just out of curiosity. He finds an interesting relation: the distance C around the circle is always the same number times the radius r – approximately 6.283… (i.e. 2π) – no matter the size of the circle.

$$C = \underbrace{2\pi}_{\approx 6.283\ldots} \cdot r$$

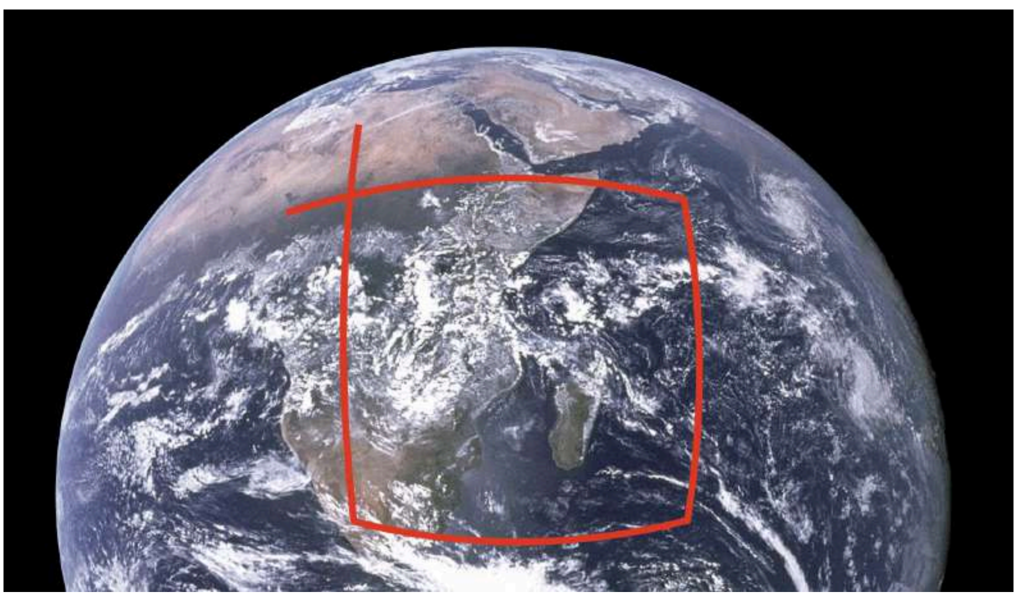
What about the other bugs? Well, the bug on the sphere tried to make a “square”, but ran into some technical difficulties… He started at a point A and drew what he calls a straight line in any direction. Then, he turns 90° and makes another straight line with the same length. Continuing this process will not allow him to “close” the square ending exactly where he started, at point A. If you do not believe me, get yourself a sphere and try it out!



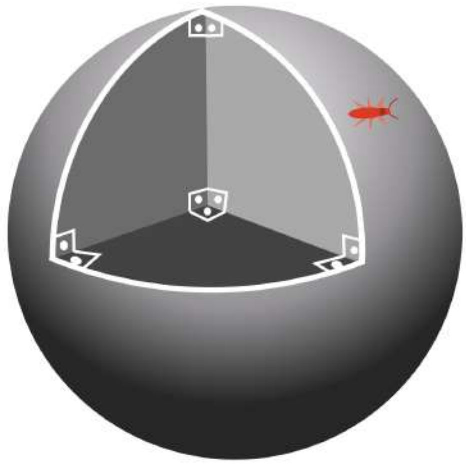
The bug in the plate will have a similar problem, but for a different reason. Since his rulers expand or contract depending on the direction of his motion (if going from hot to cold or from cold to hot), he will not be able to “close the square”.



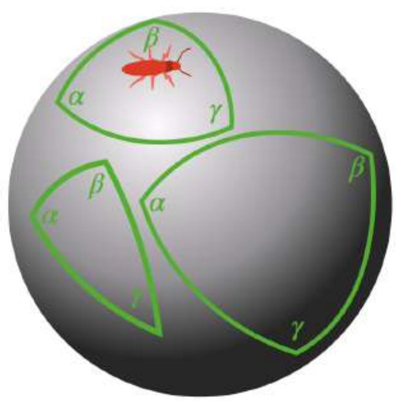
The main point here is to understand that just by geometrical measurements the second and third bugs could discover that there was something weird about their spaces. This weirdness is related to the concept of curvature. A *curved space* is a space in which the geometry is not what we expect for a plane. The last two bugs experience geometries that are characteristic of curved spaces. I.e. the rulers of Euclidean geometry fail. And you don't need to be able to lift yourself out of the surface (towards the third dimension) in order to find out that the world that you live in is curved. You do not need to go around the globe to find out that you live on the surface of a ball. All you need is to try to draw a square. If the square is small, you will need a **lot** of accuracy in order to notice that the process failed. But if the square is large enough you can easily see that.



Let's see the case of a triangle now. The bug on the sphere can construct a triangle that has 3 right angles! So weird! The bug can start at the north pole and make a straight line all the way down to the equator. Then he turns 90° and make another straight line with the same length. Then again. And there we go: a triangle with 3 right angles.

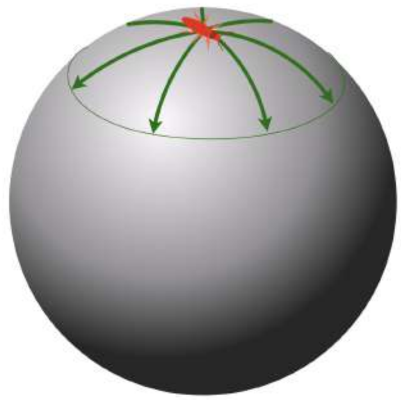


Actually, all triangles he draws will have the sum of its angles greater than 180°. If his buddy Euclidicus told him that it was supposed to add up to 180° he would probably laugh. And the bug on the plate would also laugh at this statement, but for a different reason.

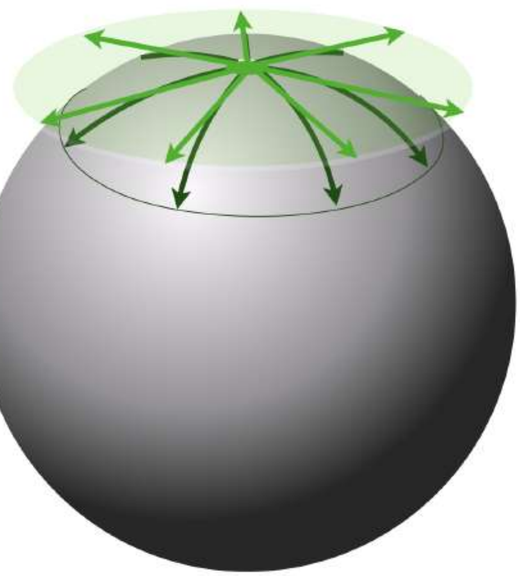


$$\alpha + \beta + \gamma > 180°$$

What about circles? How do the other bugs interpret circles in their spaces? The bug on the sphere makes a circle using the same procedure as we've seen before.



After measuring the circumference he finds out that it is *less* than $2\pi \cdot r$. For us, who are extremely privileged observers, it is obvious that a circle with this radius r would have a greater circumference in comparison to the one made by the poor 2-dimensional bug. Actually, we know that what the bug calls “radius” is a curve that is longer than what we call radius.



Now this bug talks to his friend Euclidicus, who teaches him a cool way of predicting the radius of his circle by dividing the circumference C by 2π:

$$r_{prediction} = \frac{C}{2\pi}$$

Then, in order to check if Euclidicus was legit, the bug on the sphere decides to measure the radius by himself. He would find out that the measured radius was actually larger than the predicted one. This difference is what he called "the excess radius".

$$r_{excess} := r_{measured} - r_{prediction}$$

And he went on to win the Fields medal for it.

It turns out that the excess radius is related to the size of the circle.

Meanwhile, the bug in the plate was feeling a little jealous. So he decided that he would pursue his own geometric postulates, because his space was way *cooler* – relatively speaking. He drew a circle centered at the cold spot, and what he calls "radius" are actually curves for us.

When he measures the circumference he too finds out that the measured radius is longer than the predicted radius, i.e. $\frac{C}{2\pi}$. He screams 'EUREKA', and finds out what he calls the "surplus radius".

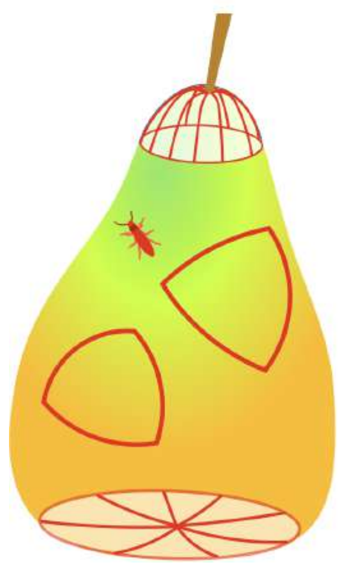
$$r_{surplus} := r_{measured} - r_{prediction}$$

He tried to publish a paper about it, but it was too late for him…

A *curved space* is one in which these kinds of errors occur: the sum of the angles of a triangle is different from 180°, the circumference of a circle divided by 2π is not equal to the radius, the rule for making a square does not produce a closed figure. You can think of others… Let us know: dibeos.contact@gmail.com. We are curious to know about other ways of noticing curvature of a space without mentioning an external dimension.

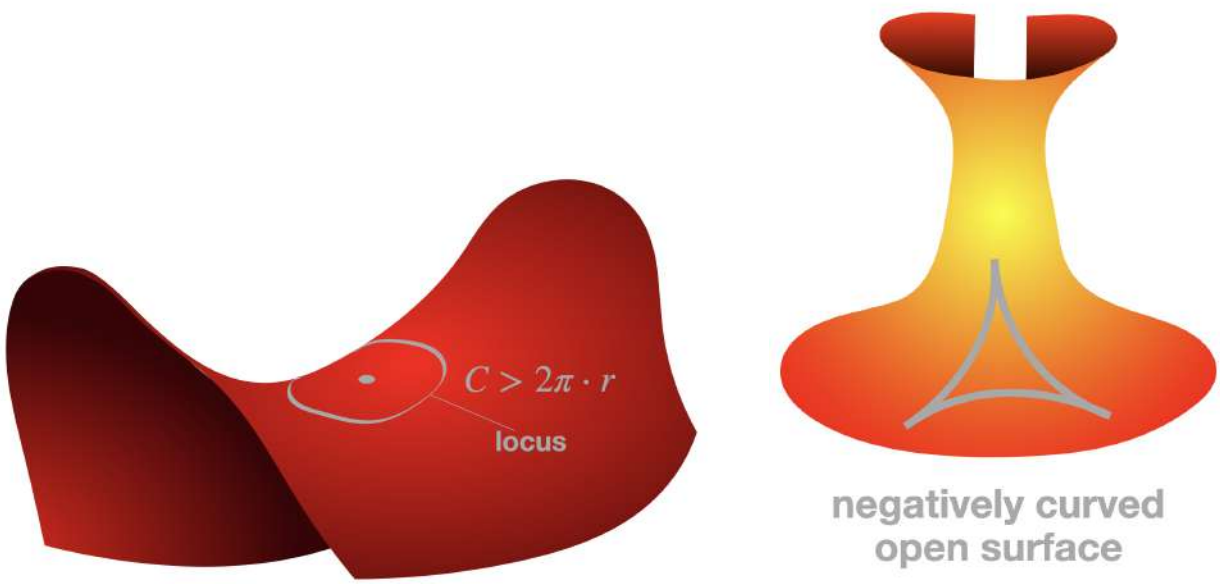
These are just two examples of curved spaces. In the plate, if you choose the right temperature variation as a function of distance (so the right scalar field), then its geometry can be identical to the sphere. I.e. as a bug living in such a space you couldn't really tell if you lived on a sphere or on a plate with different temperatures! Crazy, huh?!?!

There are, of course, different kinds of geometry. What about a bug living on a pear? At some points this space presents sharp curvatures. At others it is almost flat. Triangles here would also have the sum of their angles *greater* than 180°, no matter where you draw it.



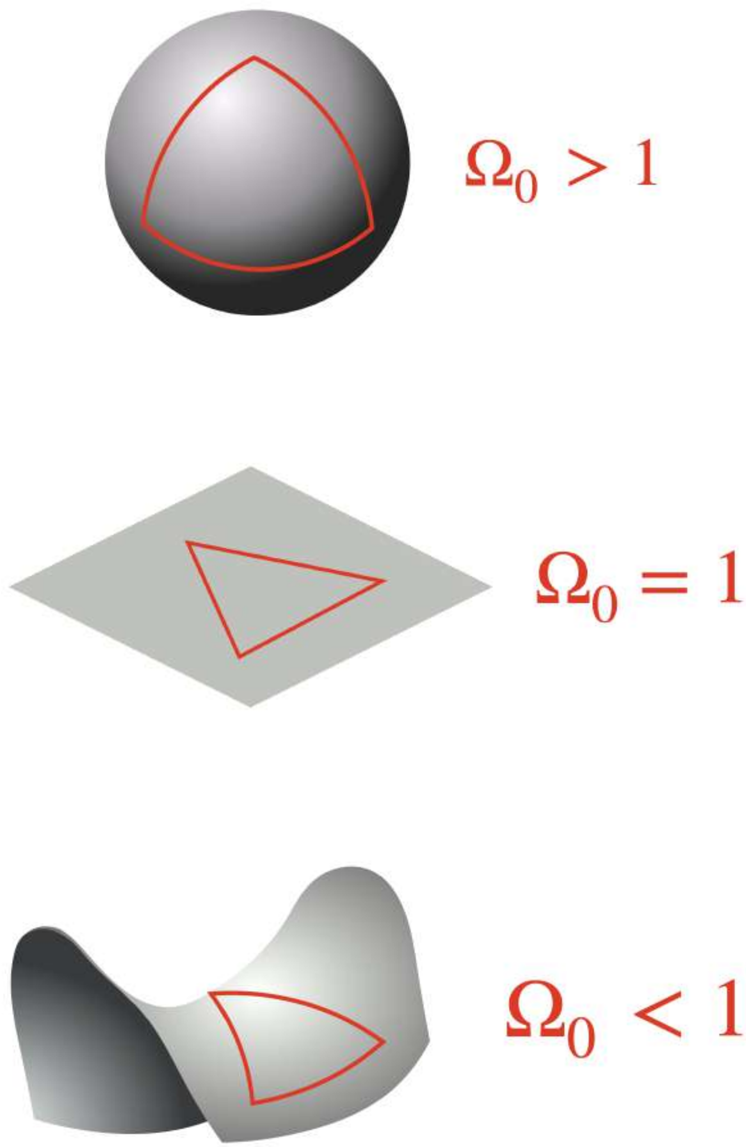
A question that may have come to your mind is: is it possible to find a triangle in a space with the sum of its angles being *less* than 180°? Yes, but not in the spaces we've seen so far.

The sum of the angles of a triangle is greater than 180° on a *positively curved*, *closed* surface (such as a sphere and a pear). Conversely, it is less than 180° on a *negatively curved*, *open* surface (such as a hyperbolic plane). Imagine a saddle, for example. Now, draw a “circle” on the surface. This is called the *locus* of all points at the same distance from the center. Its circumference is actually *larger* than you would expect from calculating $2\pi \cdot r$. In this case, the predicted radius $r_{predicted} = \frac{C}{2\pi}$ would be less than the measure radius r. And as a consequence, the “excess radius” would be negative.



General Relativity predicts 3 possible geometries for the universe, determined by the *density parameter*, Ω_0 , which compares the actual density of the universe to a critical density. If $\Omega_0 > 1$ the universe has positive curvature. This type of universe is finite and closed. So, if you traveled far enough in one direction, you would eventually return to your starting point. Over time, the gravitational pull of matter would cause the expansion of such a universe to stop and reverse direction, which would lead to a catastrophic collapse known as the "Big Crunch." On the other hand, if $\Omega_0 = 1$, the universe is flat, with zero curvature, and thus the familiar rules of Euclidean geometry can be used to study its structure in large scales. This flat universe would expand forever, but the rate of expansion would asymptotically approach zero as gravitational forces gradually slow the outward motion.

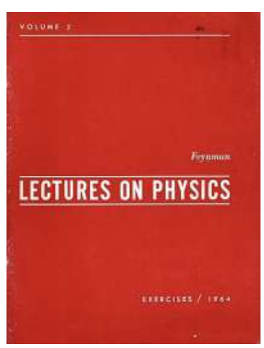
If $\Omega_0 < 1$, the universe has *negative curvature* and is *infinite*. In this case, the universe would also expand forever, but at an ever-increasing rate because of the insufficient gravitational pull to counteract the initial expansion.



Physicists are still uncertain which model accurately describes our cosmos. The problem is that it is really hard to precisely measure the density of matter and energy in the universe, especially dark energy, which is exactly what drives the acceleration of expansion.

Going back to our bugs, we notice that from the definition of curvature that we've seen so far, a cylinder is *not curved*. If a bug lives on a cylinder, then triangles, squares, and circles would all have the same behaviour they have on a plane. I mean, all you need to do is unroll the cylinder onto a plane, and you will find that out. And therefore, there is no way for a bug living on a cylinder to discover that his space is not a plane, unless he goes all the way around it, because then he would notice that he came back to the starting point after moving on what he calls a "straight line". So, in our technical sense, we consider that his space is **not** curved. What we are really talking about, though, is the concept of *intrinsic curvature*, which is crucial for the whole understanding of Differential Geometry, and as a consequence of General Relativity.

This document was based on the last chapter of the book [“The Feynman Lectures on Physics” by Feynman, Leighton, Sands – Volume II]:



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