How to Learn Analysis Effectively

by DiBeos

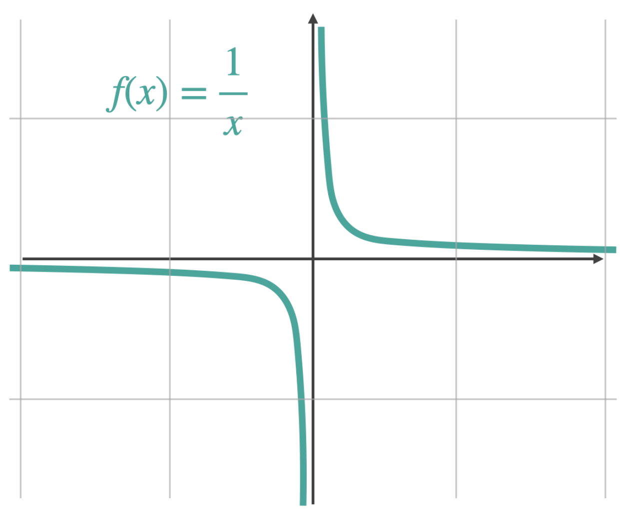
The very first thing to do when learning anything (not only Analysis) is to start with *intuition*. You know, there will be 3 steps here.

---

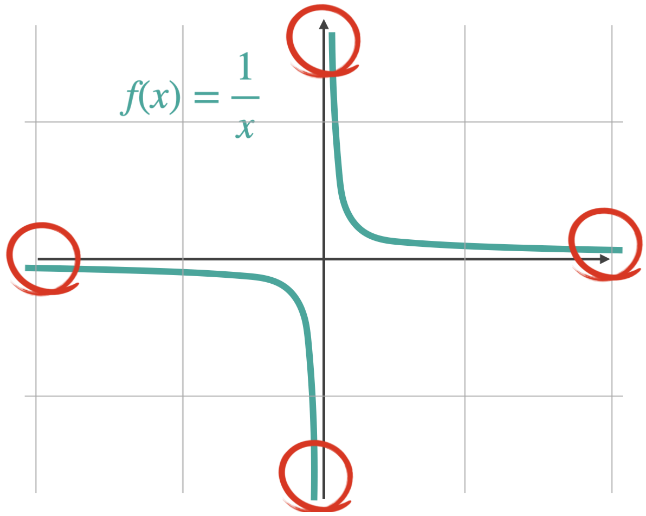
1. *Intuition*
2. *Abstraction*
3. *Practice*

The second and third are more specific to Analysis, but the first one is just a rule of thumb. When I say *intuition* I mean a non-rigorous, or even "sloppy" explanation of the concepts that you are trying to learn. Let's see an illustration.

Say you wanna learn what the *limit of a function* is. So, pick the function $f(x) = \frac{1}{x}$, for example. Its graph looks like this:



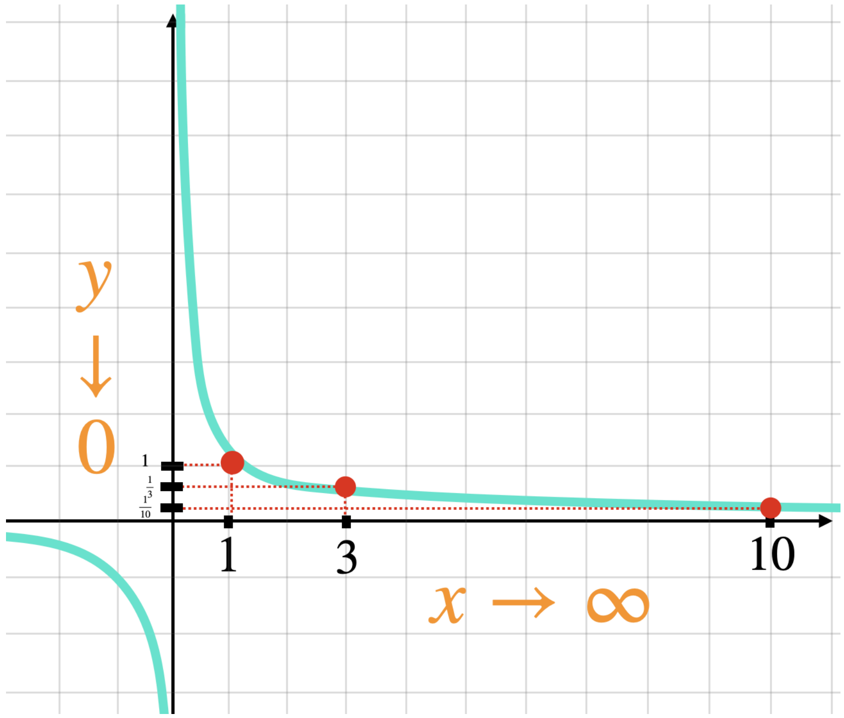
Notice how the curve gets closer and closer to the x-axis in the right region of the graph, without ever touching or crossing it. A similar thing happens for the other parts of the graph (top, bottom and left regions). The function gets closer and closer to the axes without ever touching or crossing them.



Intuitively, we can study how the points in the vertical y-axis behave as we move towards the right (so, to $+\infty$) in the x-axis. Of course, the relation between the points in the x-axis and in the y-axis is conditioned by the function $f(x) = \frac{1}{x}$. And we express it this way:

$$\lim_{x \to +\infty} \frac{1}{x}$$

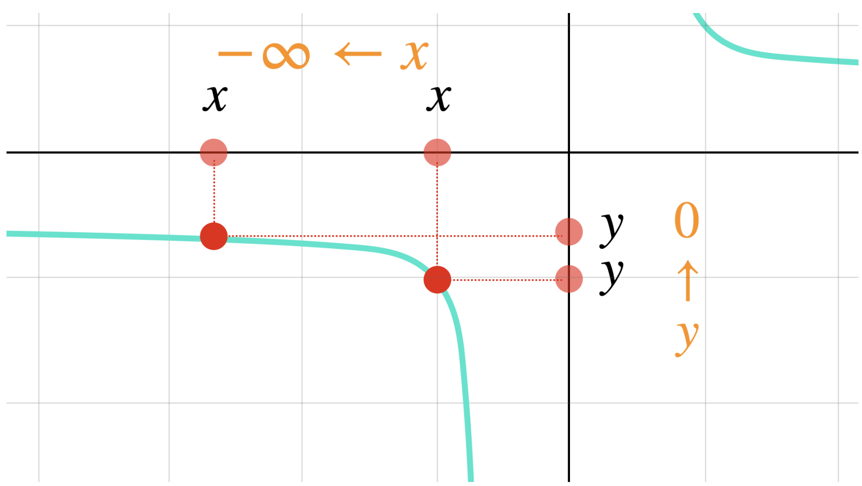
Let's take the point $x = 1$. We see that its value in y is $\frac{1}{1} = 1$. Since we need to make $x \to +\infty$ we pick the next point on the right side of $x = 1$. Let's try $x = 3$. Now $y = \frac{1}{3}$. For $x = 10$, $y = \frac{1}{10}$, and so on. We notice a pattern here. The more we move towards the right in the x-axis, the more we end up moving down in the y-axis, towards zero.



Interesting! So, we can conclude that, when $x \to +\ \infty$, $y \to 0$. In other words,

$$\lim_{x \to +\infty} \frac{1}{x} = 0$$

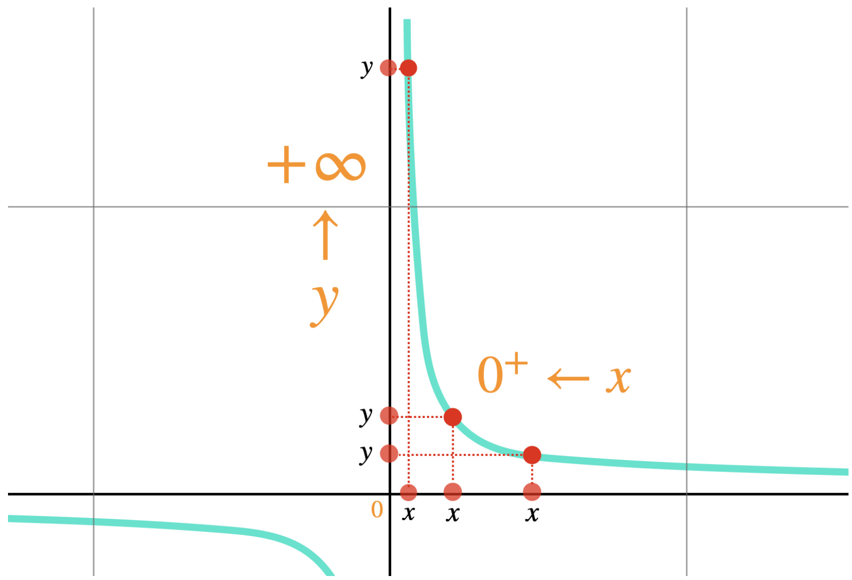
Let's see what happens for $x \to -\ \infty$. Pick a point in x and we get a point $f(x)$ in y. So, the more we move to the left in x, the more we move up in y, towards zero again.



The conclusion is:

$$\lim_{x \to -\infty} \frac{1}{x} = 0$$

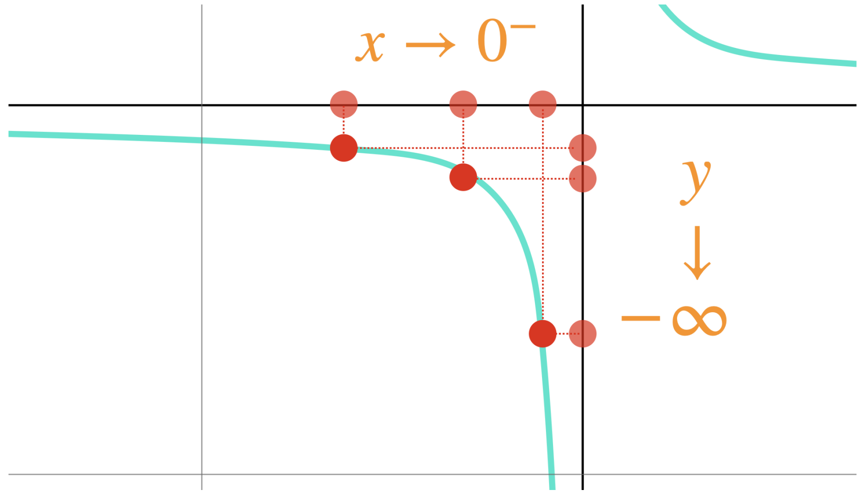
Now we analyse what happens in the top region of the graph. If we pick a point in $x > 0$, then there is a corresponding point in $y > 0$. We move x closer to zero, and y moves up. We notice that the closer we get to zero in the x-axis, the higher is the value of y, without any boundary.



In other words:

$$\lim_{x \to 0} \frac{1}{x} =+ \infty$$

Actually, this equation is <u>not</u> completely right when expressed this way. In order to understand what I mean by that, let's check out the function's behavior at the lower part of the graph. As $x \to 0$ now, $y \to -\infty$.



So, how is it possible that

$$-\infty = \lim_{x \to 0} \frac{1}{x} = +\infty \text{ ????}$$

Simple, it isn't. In fact, one of the most basic theorems in Analysis is the one that says that the limit at a point must be unique. So, since there is an ambiguity here, we say that the limit of the function at this point does not exist:

$$\nexists \lim_{x \to 0} \frac{1}{x}$$

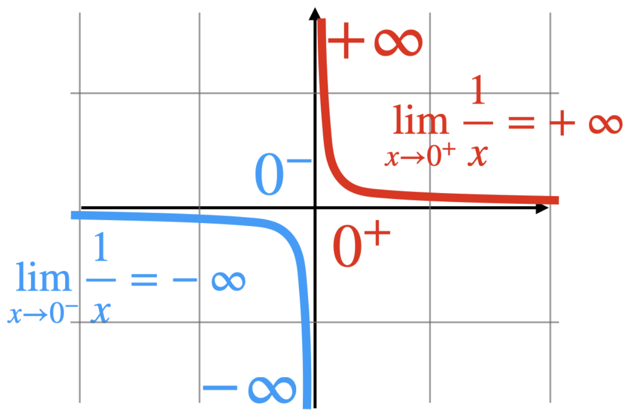
Instead, a more appropriate expression would be the following:

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

With a plus sign on top of 0 so that we know that x tends to zero from the right side (the positive side).

Meanwhile,

$$\lim_{x\to 0^-} \frac{1}{x} =- \infty$$



I don’t know if you can remember yourself when you were just learning these concepts, or if you are literally learning them for the first time right now, but this explanation, in my opinion, very clearly illustrates what the limit of a function is, why we calculate it, and what we can expect the result to look like. After creating this solid intuition behind the concept of a limit, you, not only can, but MUST move to the second step, which is to reshape it in an *abstract/general* way.

Analysis is heavily proof-based. So, eventually you do have to “mature” from just the intuition to a deeper understanding of definitions, with all of their rigor. My advice is: don't just memorize definitions; you need to DISSECT them. You need to ask yourself:

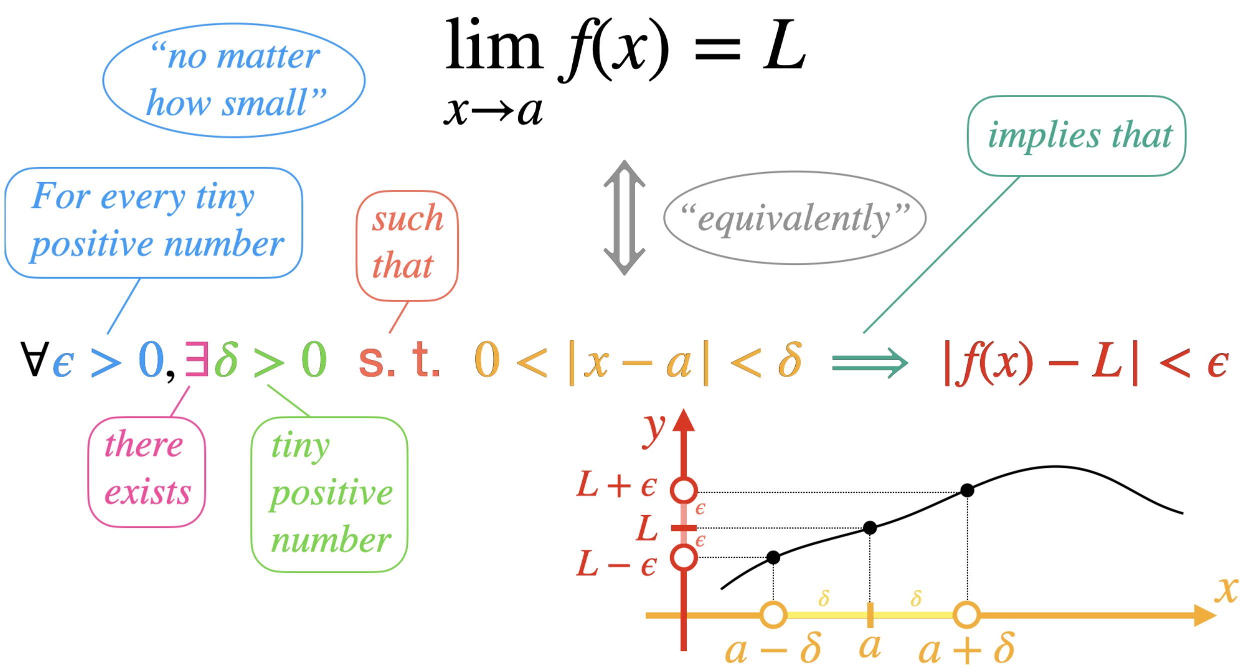
- Why is each condition necessary?
- What happens if I remove or alter a part of the definition?

In the beginning it will be time consuming, but I guarantee you that eventually it will pay-off and you will become exponentially better at it. Reading proofs will be as easy as reading any other book. But you need to pay the initial price of *dissecting* them first. Especially when studying proofs, you need to identify the key ideas first, like why a particular technique is chosen. I like to think about proofs as a narrative rather than just a mechanical process. People can have different opinions about it, but I find it very useful to study the proof of a theorem almost like a story that is being told to me. With all of its "chronological events" in order. This way you can clearly understand the reason behind every calculation or definition that is introduced in the middle of it, and also it helps you to keep track of the final goal we are trying to achieve, i.e. keep in mind what you wanted to prove in the first place.

Now that we have a good feeling of what the limit of a function is, and most importantly, of what our goal is when calculating limits, let's see its formal definition in terms of ϵ and δ:

This will be written in mathematical language, of course, but I will "translate" it to English:

'*For every tiny positive number* ϵ *, no matter how small, there exists another tiny positive number* δ *, such that whenever* x *is within the distance* δ *from 'a' (but not equal to* a*), the value of* $f(x)$ *is within the distance* ϵ *from L.*'



Take your time to digest it.

This is actually the particular case in which x tends to a finite number a , and the result of the limit is finite as well – the value L here . There are other 3 cases for the definition of the limit of a function as well. They are the following: (the first definition is the one we just saw)

Definition 1: (a finite and L finite) $\boxed{\lim_{x\to a} f(x) = L}$

$$\forall \epsilon > 0, \exists \delta > 0 \text{ s. t. } 0 < |x - a| < \delta \implies |f(x) - L| < \epsilon$$

Definition 2: (a finite and $L = +\infty$) $\boxed{\lim_{x\to a} f(x) = +\infty}$

$$\forall M > 0, \exists \delta > 0 \text{ s. t. } 0 < |x - a| < \delta \implies f(x) > M$$

large positive number

Definition 3: ($a = +\infty$ and L finite) $\boxed{\lim_{x\to +\infty} f(x) = L}$

$$\forall \epsilon > 0, \exists N > 0 \text{ s. t. } x > N \implies |f(x) - L| < \epsilon$$

large positive number

Definition 4: ($a = +\infty$ and $L = +\infty$) $\boxed{\lim_{x\to +\infty} f(x) = +\infty}$

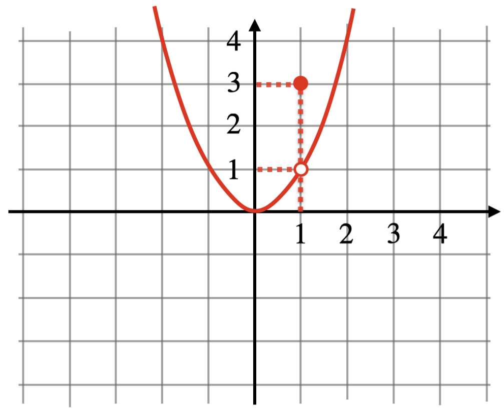
$$\forall M > 0, \exists N > 0 \text{ s. t. } x > N \implies f(x) > M$$

large positive number *large positive number*

Notice though that just because the limit of a function, with x that tends to a finite value a , is finite (so, L) – first definition – , it does not mean that the function itself is *continuous* at that point, i.e. it does not mean that $f(a) = L$. Let's see a counterexample:

Consider the function

$$f(x) = \begin{cases} x^2 & \text{if } x \neq 1 \\ 3 & \text{if } x = 1 \end{cases}$$



The limit as $x \to 1$ is

$$\lim_{x \to 1} f(x) = 1^2 = 1$$

This is true for the limit to the right and to the left of $x = 1$, so this limit does exist. However, when we look for the value of the function exactly at the point $x = 1$, we find out that

$$f(1) = 3, \text{ so } \lim_{x \to 1} f(x) \neq f(1)$$

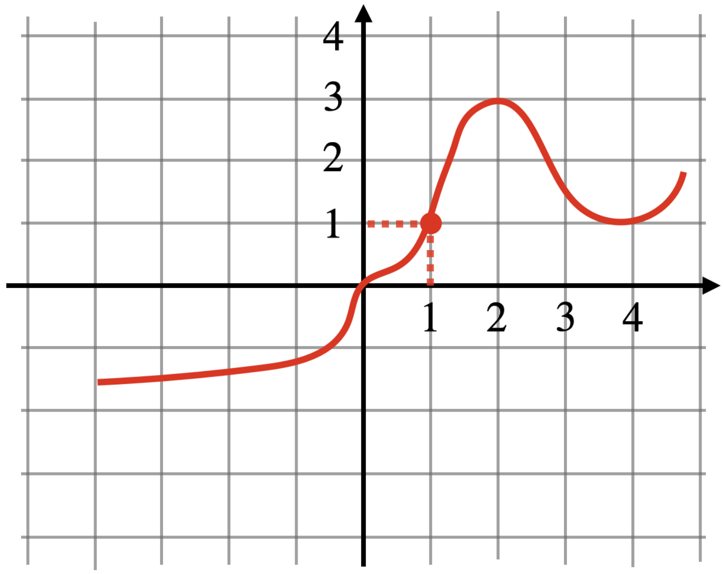
$$f(x) \text{ is not continuous at } x = 1$$

In fact, the rigorous definition of a *continuous function* is not the same as the definition of the *limit* of a function. Let's see the definition, using ϵ and δ, of a continuous function:

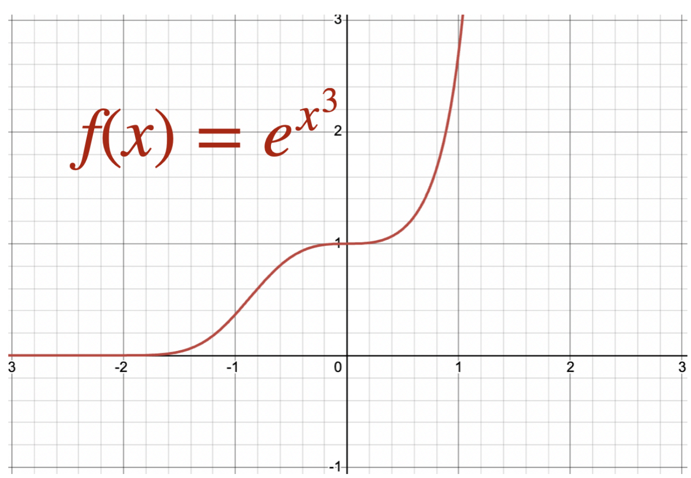
Let $f : D \to \mathbb{R}$, and let $c \in D$. The function f is continuous at the point c if:

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ : \ |x - c| < \delta \implies |f(x) - f(c)| < \varepsilon, \ \forall x \in D.$$

An example of function f continuous at $x = 1$.



Let's see a quick example: $f(x) = e^{x^3}$ is continuous for all real numbers.



Pick any small positive number ϵ . Then, there will always be another small number δ such that for any point $x \in (-1 - \delta, -1 + \delta)$ – so, we will prove continuity for the point $x = -1$, in this case – we have that $e^{x^3} \in \left(e^{(-1)^3} - \epsilon \ , \ e^{(-1)^3} + \epsilon \right)$. This implies that $e^{x^3} \in \left(\frac{1}{e} - \epsilon \ , \ \frac{1}{e} + \epsilon \right)$.

Let's do it in a more concrete way:

Pick any small positive number, say $\epsilon := 0.1$. Then, there will always be another small number δ (we need to find the value of δ that will work fine with this particular choice of ϵ) such that for any point $x \in (-1 - \delta, -1 + \delta)$ – so, we will prove continuity for the point $x = -1$, in this case – we have that $e^{x^3} \in \left(e^{(-1)^3} - 0.1 \ , \ e^{(-1)^3} + 0.1 \right)$. This implies that $e^{x^3} \in \left(\frac{1}{e} - 0.1 \ , \ \frac{1}{e} + 0.1 \right)$.

Let's find such δ :

$$\left| f(x) - f(-1) \right| < \epsilon \implies \left| e^{x^3} - e^{(-1)^3} \right| < 0.1 \implies$$

$$\implies \left| e^{x^3} - \frac{1}{e} \right| < 0.1 \implies -0.1 < e^{x^3} - \frac{1}{e} < 0.1 \implies$$

$$\implies -0.1 + \frac{1}{e} < e^{x^3} < 0.1 + \frac{1}{e} \implies$$

$$\implies \ln\left(-0.1 + \frac{1}{e} \right) < x^3 < \ln\left(0.1 + \frac{1}{e} \right)$$

We know that $x^3 = (x+1)^3 - 3(x+1)^2 + 3(x+1) - 1$. For small deviations around $x = -1$, we approximate:

$$\boxed{x^3 \approx -1 + 3(x+1)}$$

$$\ln\left(-0.1 + \frac{1}{e}\right) < x^3 < \ln\left(0.1 + \frac{1}{e}\right) \implies$$

$$\implies \ln\left(-0.1 + \frac{1}{e}\right) < -1 + 3(x+1) < \ln\left(0.1 + \frac{1}{e}\right)$$

$$\implies \frac{\ln\left(-0.1 + \frac{1}{e}\right) + 1}{3} < x - (-1) < \frac{\ln\left(0.1 + \frac{1}{e}\right) + 1}{3}$$

$$\therefore \quad \delta = \min\left(\frac{\ln\left(0.1 + \frac{1}{e}\right) + 1}{3}, -\frac{\ln\left(-0.1 + \frac{1}{e}\right) + 1}{3}\right)$$

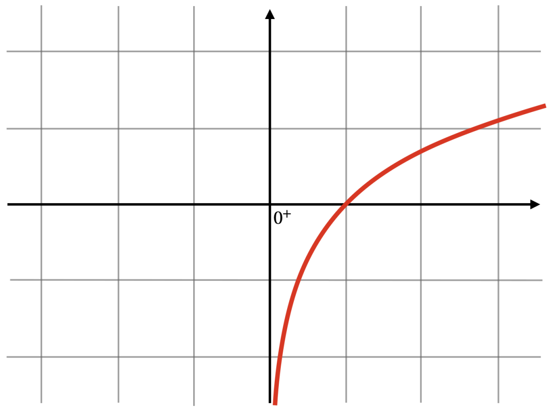
(Try to create numerical examples on your own and check how this definition of continuity is indeed consistent)

So far we saw *intuition*, and *abstraction*. Now it's time to move to the last step in the process of learning Analysis: *practice*. What do I mean by that? I mean actually creating concrete examples of your own, where you can perform practical calculations to see if your intuition and understanding of the rigorous definitions and theorems align with problems involving numbers. Let's illustrate it with

$$\lim_{x \to 0^+} \ln(x) = -\infty$$

The graph of $f(x) = \ln(x)$ looks like this (see below). And we can clearly see how the function, indeed, tends to $-\infty$ when $x \to 0^{+}$. Notice that the function has no values for $x < 0$.

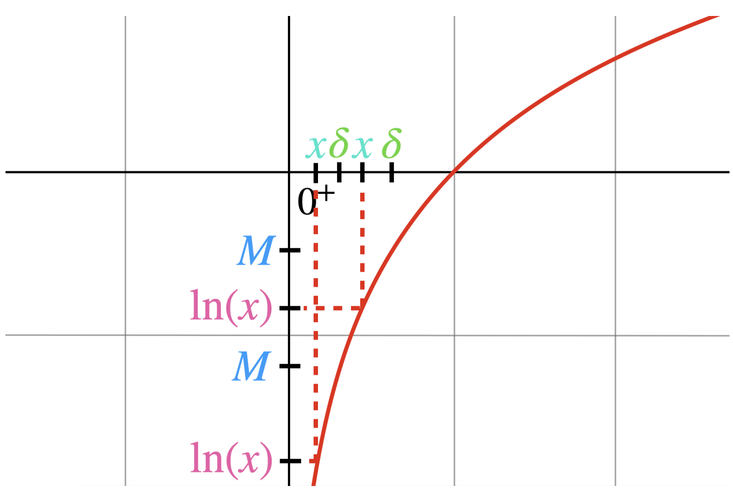
$$\lim_{x \to 0^{+}} \ln(x) = -\infty$$



Its rigorous definition is the following:

$$\forall M < 0 \,,\, \exists \delta > 0 : 0 < x < \delta \Rightarrow \ln(x) < M.$$

Let's see its behavior in the graph (see below). Pick any negative value M in the vertical axis. No matter how "low" it is, there is always a possible choice of a tiny δ such that you can find a value x in the horizontal axis that gives us a value $\ln(x)$ even lower than the previously chosen M. Again, no matter how low M is, you can always find a point $\ln(x)$ lower than that.



Of course, looking at the graph it is kind of an obvious fact, but we need to show that *analytically*, i.e. *rigorously*.

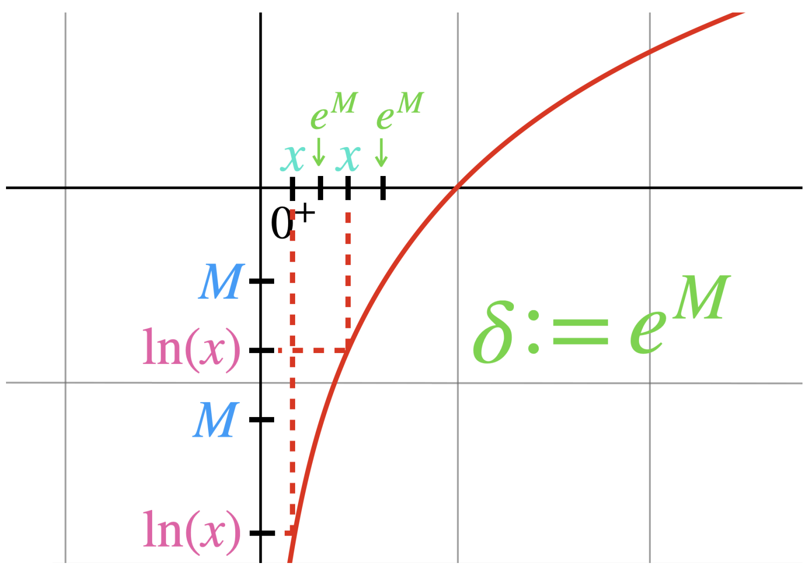
For every $M < 0$, our goal is to find a $\delta > 0$ such that whenever $0 < x < \delta$, we also have (as a consequence) that $\ln(x) < M$. So, at the end of the day $\ln(x) < M$ is what we want to prove, and so we should focus our attention on it!

$$\ln(x) < M \Rightarrow \textit{(exponentiating both sides)}\ e^{\ln(x)} < e^{M} \Rightarrow x < e^{M} .$$

So, what δ should we pick for each value of M ? What about $\delta := e^{M}$? This is not the only possible choice, but we will see now that this choice does work fine:

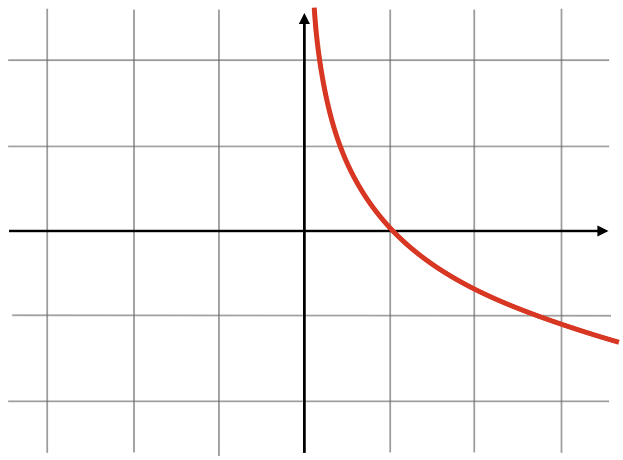
$$\forall M < 0,\ \exists\delta := e^{M} :\ 0 < x < e^{M} \Rightarrow x < e^{M} \Rightarrow \textit{(taking } \ln \textit{ in both sides)}\ \ln(x) < M$$

So, indeed, $\delta := e^{M}$ is one of the right choices that guarantees that no matter how low the value of M is, there is always a δ (small enough) that lets us find a point below M in the vertical axis.



Notice that if instead the function were $f(x) = -\ln(x)$,

$f(x) = -\ln(x)$



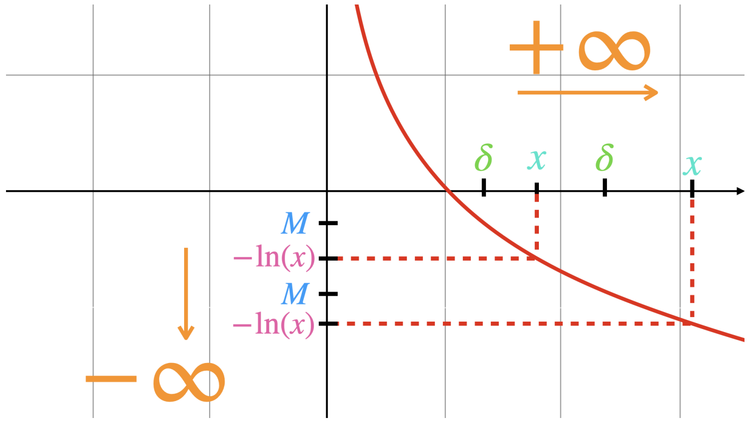
then this argument would fail. What do I mean by that? Let's try to prove the same thing for this new function:

$$\forall M < 0\,,\ \exists\delta > 0 :\ 0 < x < \delta \Rightarrow -\ln(x) < M\,.$$

This is not possible.
Once again, we focus on what we actually want to prove: $-\ln(x) < M$.

$$-\ln(x) < M \Rightarrow \ln(x) > -M \Rightarrow e^{\ln(x)} > e^{-M} \Rightarrow x > \frac{1}{e^{M}}$$

So now, $x > (something)$, which is the opposite of what we got in the previous example, which was $x < (something\ else)$. This means that we cannot find *smaller and smaller* values of δ that satisfy any choice of M . Instead, we need to find *larger and larger* values of δ to do so.



We actually found that

$$\lim_{x \to +\infty} (-\ln(x)) = -\infty$$

This last step is so important! I can't emphasize enough how important it is to practice with concrete calculations that you personally came up with. It really helps you to solidify the concepts. And it is also strongly supported by neuroscience.

Our brains are malleable, even after we become adults, which means that connections in our brains rewire themselves during learning. Basically, we physically alter the connections inside of our brain and make a structural change when we're learning. But, an important term to remember when learning is *cognitive load*. In order to properly process the information, you have to "organize, contrast, and compare" the ideas. So the more you focus on problem solving, the better you will retain the information.

In other words, after grasping the intuition and the abstract rigor of a specific concept in Analysis, you need to practice it with concrete problems over and over again. And this is the best way of learning Analysis.

This content was a little different from what we usually do, so please let us know if you guys enjoyed this mix of personal advice on how to learn specific subjects in mathematics and mathematical physics, together with some technical examples along the way.

Please, if you find this document useful, let us know. Or if you found typos and things to improve, let us know as well. Your feedback is very important to us. We're working hard to deliver the best material possible. Contact us at: dibeos.contact@gmail.com

Neurology Sources:

https://pmc.ncbi.nlm.nih.gov/articles/PMC5903872/

Bennett, S.H., Kirby, A.J. and Finnerty, G.T. (2018) *Rewiring the connectome: Evidence and effects*, *Neuroscience and biobehavioral reviews*. Available at: https://pmc.ncbi.nlm.nih.gov/articles/PMC5903872/.

https://www.sciencedirect.com/science/article/abs/pii/B9780128018729000065

Reese, D.D., Pawluk, D.T.V. and Taylor, C.R. (2016) ‘Engaging learners through rational design of multisensory effects’, *Emotions, Technology, and Design*, pp. 103–127. doi:10.1016/b978-0-12-801872-9.00006-5.