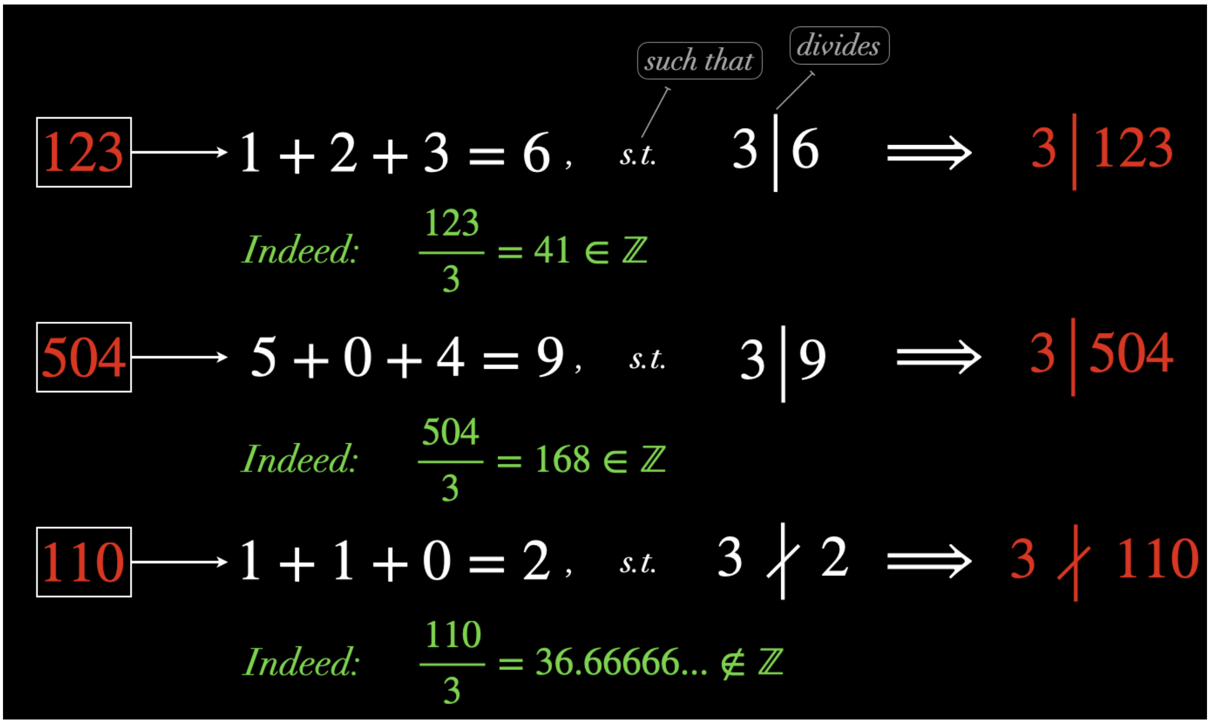
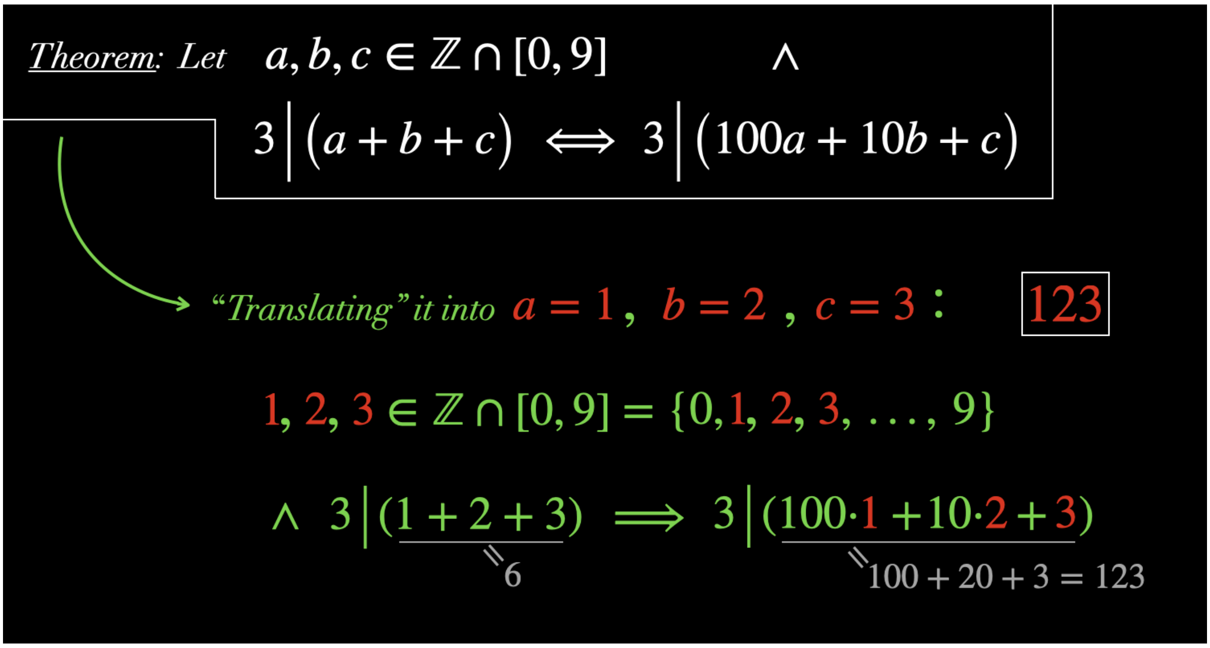
Divisibility by 3 or 9

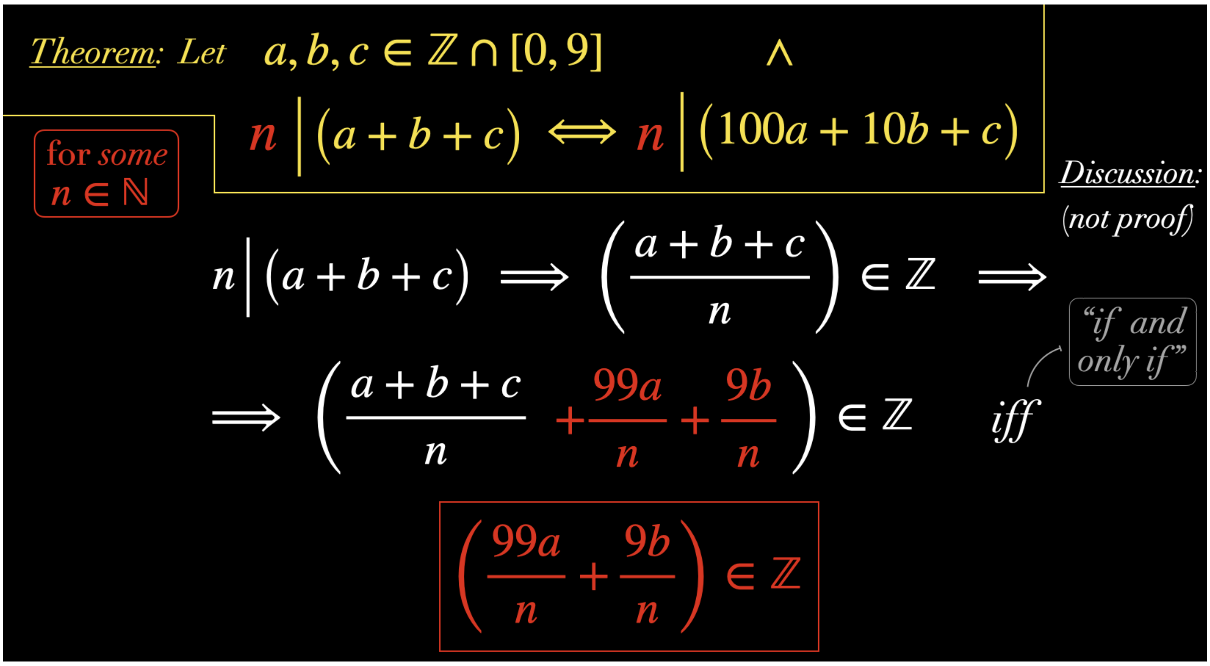
Take the numbers 123, 504 and 110. Are these numbers divisible by 3? Well, there is a clever way of checking it.



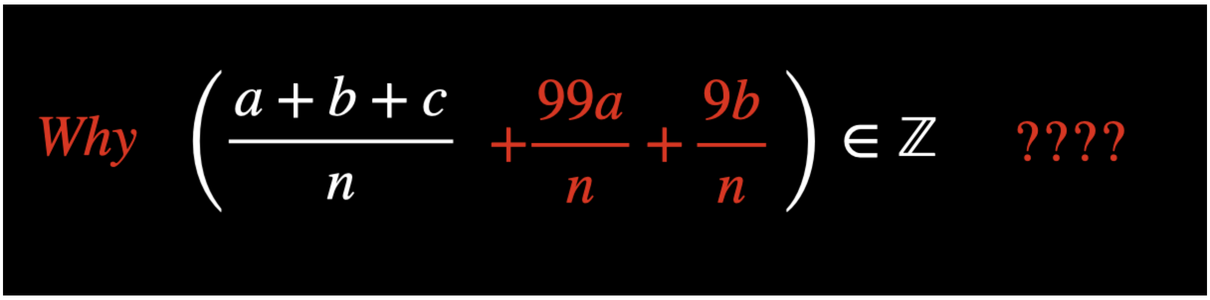
Let us generalize this result.



In order to prove this theorem (*n*=3), we will first consider an even higher level of abstraction (generalization), in which this rule is valid for some natural number *n*.



Before moving on, I have a question: Why did I do it? I.e. why would I want to impose that?



Well, because if I require that this sum is an integer, I will actually have required the following:

(BTW, consider becoming a member of the channel!) Thanks!

$$\left(\frac{a+b+c}{n}+\frac{99a}{n}+\frac{9b}{n}\right)\in\mathbb{Z}$$

$$\Downarrow$$

$$\boxed{\frac{100a+10b+c}{n}\in\mathbb{Z}}$$

In other words, I will have imposed that *n* divides *100a+10b+c*, which is exactly the implication of this theorem! (See below)

Theorem: Let $a,b,c\in\mathbb{Z}\cap[0,9]\quad\wedge$

$$n\,\Big|\,(a+b+c)\iff n\,\Big|\,(100a+10b+c)$$

for *some* $n\in\mathbb{N}$

Ok, let us pick up where we left off.

$$\left(\frac{99a}{n}+\frac{9b}{n}\right)\in\mathbb{Z} \implies \frac{9\cdot(11a+b)}{n}\in\mathbb{Z} \;\not\Rightarrow$$

$$\not\Rightarrow \begin{cases} n\,|\,9 \\ \vee / \wedge \\ n\,\big|\,(11a+b) \end{cases}$$

We cannot say that this sum implies that *n* divides *9* or that *n* divides *11a+b*, but we know that **when** *n* divides *9* this sum is indeed an integer. So, we found out that when *n=3* or *n=9* the theorem holds!

$$\left(\frac{99a}{n}+\frac{9b}{n}\right)\in\mathbb{Z} \implies \frac{9\cdot(11a+b)}{n}\in\mathbb{Z}$$

One of many possibilities:

$$n\,\big|\,9 \iff \boxed{n=3} \vee \boxed{n=9}$$

Indeed, this theorem is valid **only** for *3* and *9* (in the base-10 system)!

Just out of curiosity, let us try it out for *n=9*:

$\boxed{n = 9}$:

$\boxed{918} \longrightarrow 9 + 1 + 8 = 18$, s.t. $9 \mid 18 \implies 9 \mid 918$

Indeed: $\dfrac{918}{9} = 102 \in \mathbb{Z}$

$\boxed{109} \longrightarrow 1 + 0 + 9 = 10$, s.t. $9 \nmid 10 \implies 9 \nmid 109$

Indeed: $\dfrac{109}{9} = 12.111111... \notin \mathbb{Z}$

$\boxed{117} \longrightarrow 1 + 1 + 7 = 9$, s.t. $9 \mid 9 \implies 9 \mid 117$

Indeed: $\dfrac{117}{9} = 13 \in \mathbb{Z}$

What else can we do?
One thing is to generalize it to a number with *m* digits.

For example: $344201...4472$

$$n \mid (3 + 4 + 4 + \ldots + 7 + 2)$$

$$\Downarrow ?$$

$$n \mid (344201...4472)$$

for *some* $n \in \mathbb{N}$

Let us consider a general number denoted as $a_1 \cdots a_m \in \mathbb{Z}$:

For example: $a_1 a_2 \ldots a_m \in \mathbb{Z}$

$$s.t.\ \ a_i \in \mathbb{Z} \cap [0,9],\ \forall i \in \{\overline{1,m}\}$$

$$\wedge\ n \,\Bigg|\, \left(\sum_{i=1}^{m} a_i\right) \quad \overset{?}{\Longrightarrow}$$

$$\overset{?}{\Longrightarrow} \quad n \,\big|\, (a_1 \ldots a_m)$$

for *some* $n \in \mathbb{N}$

What are the conditions, such that $n \,\big|\, (a_1 \ldots a_m)$?
Try to work on finding these conditions by yourself, and leave a comment about what you found out. Maybe try to do something similar to what we have done before.

I will show you a way of thinking about it:

$$n \,\Big|\, (a_1 \dots a_m) \implies \left(\frac{a_1 \dots a_m}{n}\right) \in \mathbb{Z} \iff$$

$$\iff \left(\frac{10^{m-1}a_1 + 10^{m-2}a_2 + \dots + \overbrace{10^0}^{10^{m-m}} a_m}{n}\right) \in \mathbb{Z}$$

$$\iff \left(\sum_{i=1}^{m} \frac{10^{m-i}}{n} a_i\right) \in \mathbb{Z}$$

(I want to use that)

$$\iff \left(\frac{10^{m-1}a_1 + 10^{m-2}a_2 + \dots + 10^0 a_m}{n}\right) \in \mathbb{Z}$$

$$\iff \left(\frac{10^{m-1}}{n}a_1 + \frac{10^{m-2}}{n}a_2 + \dots + \frac{10^{m-(m-1)}}{n}a_{m-1} + \frac{10^{m-m}}{n}a_m\right) \in \mathbb{Z}$$

$$\left(\frac{10^{m-1}}{n}a_1 + \frac{10^{m-2}}{n}a_2 + \ldots + \frac{10^{m-(m-1)}}{n}a_{m-1} + \frac{10^{m-m}}{n}a_m\right) \in \mathbb{Z} \iff$$

$$\iff \left(\frac{a_1}{n} + \frac{\left(10^{m-1}-1\right)}{n}a_1 + \frac{a_2}{n} + \frac{\left(10^{m-2}-1\right)}{n}a_2 + \cdots\right.$$

$$\left.\ldots + \frac{a_{m-1}}{n} + \frac{\left(10^{m-(m-1)}-1\right)}{n}a_{m-1} + \frac{a_m}{n} + \frac{\left(10^{m-m}-1\right)}{n}a_m\right) \in \mathbb{Z}$$

$$\iff \left(\sum_{i=1}^{m}\frac{a_i}{n} + \frac{\left(10^{m-1}-1\right)}{n}a_1 + \ldots + \frac{\left(10^{1}-1\right)}{n}a_{m-1}\right) \in \mathbb{Z} \iff$$

$$\iff \left(\sum_{i=1}^{m}\frac{a_i}{n} + \frac{\overbrace{999\ldots9}^{m-1}}{n}a_1 + \ldots + \frac{9}{n}a_{m-1}\right) \in \mathbb{Z}$$

$$\left(\underbrace{\sum_{i=1}^{m}}_{\in \mathbb{Z}}\frac{a_i}{n} + \frac{\overbrace{999\ldots9}^{m-1}}{n}a_1 + \ldots + \frac{9}{n}a_{m-1}\right) \in \mathbb{Z} \iff$$

$$\iff \left(\frac{999\ldots9}{n}a_1 + \ldots + \frac{9}{n}a_{m-1}\right) \in \mathbb{Z} \iff \left(\sum_{i=1}^{m}\frac{9\cdot(\overbrace{1\ldots1}^{m-i})}{n}a_i\right) \in \mathbb{Z}$$

One of many possibilities:

$$n \mid 9 \iff \boxed{n=3} \lor \boxed{n=9}$$

Therefore, we found out that it does not matter how many digits an integer has, it is always true that if the sum of its digits is divisible by

3, then the number itself is also divisible by *3*. And the same statement is true for *9* as well.

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