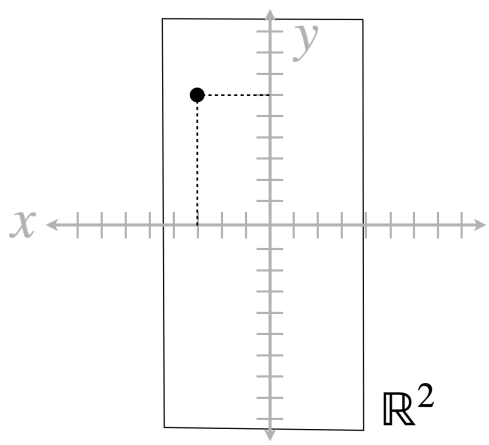
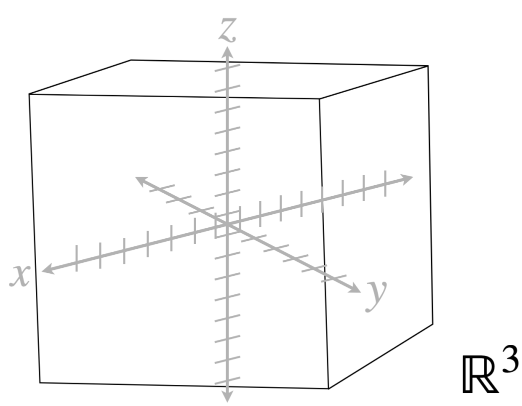
Manifolds

A room can be broken down into a combination of points, lines, which can be extended indefinitely in both directions, planes, angles and the distance between them.

But, what if we overlay Cartesian coordinates on, pretty much any surface we want in that room? Well, then we'll have a 2D representation, pretty much like a plane, but with coordinates that can be labeled x and y, which allows us to pinpoint locations on the plane—this is what we refer to as $\mathbb{R}^2$.



However, to capture all sides of an object, we move to a 3-dimensional space $\mathbb{R}^3$, where instead of having just 2 directions x and y, we have 3 (z), allowing us to describe any point with three coordinates.



The concept doesn't have to be confined to just three dimensions. We can conceptualize Euclidean spaces in higher dimensions as well. A Euclidean space can be n-dimensional, holding all possible tuples of n real numbers (x_1, x_2, ..., x_n), with each number representing a coordinate in one dimension of that space.

The distance between two points $x_1, x_2, ..., x_n$ and $y_1, y_2, ..., y_n$ in $\mathbb{R}^n$ is given by the formula $\sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + ... + (x_n - y_n)^2}$. This is a generalization of the Pythagorean theorem.

In itself, we just described what a Euclidean space is. But did you know that if we zoom out, we actually described a manifold, but locally?

The idea is that at a very small scale, every part of the loop looks just like a tiny segment of a straight line, despite the overall curved shape.

Thus, you can find a continuous one to one mapping from that straight line to a closed loop, so in other words, they are *homeomorphic*.

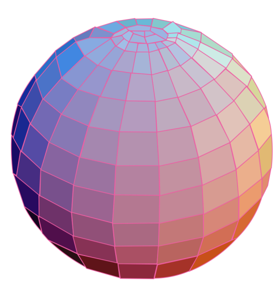
Open-ended or even infinitely extending curves like parabolas, hyperbolas, and cubic curves, still have a local structure similar to a line, making them 1D manifolds as well.

But, not all closed shapes fit under that definition. For example, the number 8 has an intersection in the middle, and this crossing point, no matter how much you zoom in on it, will not resemble a single line locally.

The same thing can be said if we increase the dimension to 2.

The simplest thing we can start with is a sphere. Locally, you can imagine that it looks like a flat plane, and again, the two are homeomorphic.

Thus, any 2 dimensional surface that doesn't self intersect is also a 2 dimensional manifold.



We can go on to higher dimensions, but they're pretty difficult to imagine. So, what can we do?

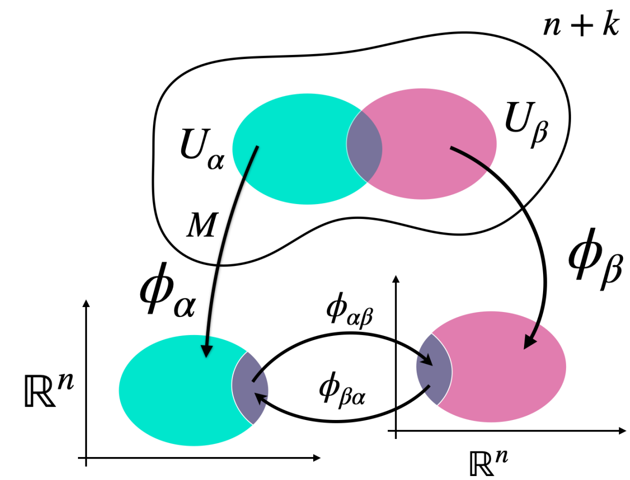
Say we have a manifold, and we call it *M*. We imagine it to be embedded in an *n+k* dimension. The n refers to the dimension of the manifold itself, while the k represents additional dimensions that are not part of the manifold but are part of the surrounding space in which the manifold is embedded.

We have 'patches' called local coordinate neighborhoods. One U_α a and U_β, and they are collections of points, or open sets.

Now remember, since *M* is a manifold, we can associate, or map, each point of one set the manifold with exactly one element of a Euclidean space $\mathbb{R}^n$.

This is done via a function, ϕ

Suppose *p* is a point in U_α. Applying the chart ϕ_α to p might convert the geographical position of *p* into a pair of numbers representing its coordinates in a 2D plane, say *(x,y)*. This mapped point is called the *coordinate* or *local coordinate* of p in our chart or coordinate system.



If we have a bunch of charts that together cover every point on a manifold, it's called an atlas.

When two charts, like ϕ_α and ϕ_β , overlap, the functions that convert coordinates from

one chart's system to another are called *transition maps*.

$\phi_{\alpha\beta} = \phi_\beta \circ \phi_\alpha^{-1}$ this function translates coordinates from the ϕ_α system to the ϕ_β system.

$\phi_{\beta\alpha} = \phi_\alpha \circ \phi_\beta^{-1}$ this one translates coordinates from the ϕ_β system back to the ϕ_α system.

Specifically, $\phi_{\alpha\beta}$ works within $\phi_{\alpha}(U_{\alpha} \cap U_{\beta})$, and $\phi_{\beta\alpha}$ within $\phi_{\beta}(U_{\alpha} \cap U_{\beta})$. These areas are where both charts provide valid coordinates for points on the manifold.

The "smoothness" or differentiability of manifolds is categorized by how many times you can continuously take derivatives of these transition maps.

Being able to continuously take derivatives up to a certain order *k* means you can compute the *1st, 2nd, ..., kth* derivatives, and each of these derivatives is a smooth function itself.

C^{0} is the simplest level of differentiability, where functions are just continuous, with no sudden jumps.

C^{1} are functions that are smooth enough to take one derivative, like going from a straight but potentially cornering path to a smoothly curving path.

C^{k} is where you can take *k* derivatives one after the other, and all these derivatives are smooth.

C^{∞} is the highest level, where you can take an infinite number of derivatives, and all are smooth.

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