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# Absence of Diffeomorphism Symmetry for Black Hole Spacetime: a prelude to Black Hole Spacetime being a Smooth Finsler Manifold

**Shreyansh Singh**

*E-mail:* shreyanshsingh25031998@gmail.com

Abstract: We propose another version of Black Hole information Paradox owing absence of Diffeomorphism symmetry for Black Hole Spacetime. The central idea is assumptions of semiclassical Gravity like locality, causality and diffeomorphism symmetry lead to another paradox. The paradox unlike many other recent versions of Information paradox is valid even at very early times.The possible suggestive way for the resolution of paradox has various approaches like validity of Black Hole Complementarity from the very begining or the principle of Holography of Information proposed by Suvrat et. al. We discuss some subtleties with these approaches in the local nature of splitting of Hilbert Space(in some pseudoprecise way) of the dual CFT. Being as conservative as we can ,to avoid such subtleties (which have been addressed in the paper) we suggest that absence of Diffeomorphism symmetry for Black Hole Spacetime implies it being a smooth Finsler manifold. We will discuss what we mean by this and the implications of it. We also give a recipe for recosntructing Null Hypersurface for Black Hole Spacetime in a different way in section 3.6, 3.7 of the paper.We are also lead to notion of Finsler Geometry for Black Hole Spacetime owing to asymmetry of line element.We also find that the Null Diffeomorphism Symmetry group of 2 dimensional Minokowski spacetime admits a superspace structure.However owing to analytical difficulty of solving coupled ordinary Raychaudhari differential equations and a consistency condition we could not verify the claim of superspace structure for Null Diffeomorphism group for Schwarschild spacetime.

## Contents

## 1 Introduction

The framework of Einstein General Theory of Relativity is significantly different from Newtonian standpoint which deals with the idea of studying classical dynamics in flat spacetime. The idea of General theory of relativity is to assume spacetime to be curved and the dynamics happens in curved spacetime, this new notion of geometry is at the crossroads of the branch of mathematics we know as Differential Geometry and topology. Why do we need the tools of semi modern mathematics to study Einstein General Theory of relativity? The simplest answer is the usual known notions , intutions of Flat spacetime like assuming a vector space structure, tangent vector may not apply in usual way in curved spacetime simply because of the curvature of spacetime means the local data assigned at a point changes as we proceed along the spacetime. This also implies that spacetime needs more primitive structure to be addressed in this situation, mathematicians while dealing with curved spaces associate the idea of a manifold.[1]
While dealing with curved spacetime we need to associate Einstein Equivalence principle and Lorentz transformation which are applicable in flat spacetime to be replaced by more generic notions which are applicable locally. The basic idea is to preserve the notion of smoothness of manifold so that it allows a geometrical structure as a manifold thereby allowing tools and techniques of a smooth manifold[2]like diffeomorphisms[3] and relatable ideas from Reimannian geometry to be applicable. Our aim through this paper will be to first address the tools and ideas from differential geometry which seem to be hidden or not properly addressed. However we will see that even notions of Black Hole Spacetime being a smooth differentiable manifold comes in conflict with the known principles of Einstein special theory of relativity when considering Black Hole Spacetimes. We try addressing these aspects in various ways, as a possibility we try being as conservative as we can (for now) and suggest that spacetime as addressed in classical general relativity is not a smooth Riemannian manifold and we suggest various possibilities like it being a Finsler Geometry and so on. Considering various alternatives like Black Hole Complementarity and Principle of Holography of Information for the possible resolution we find that motivated by Holography the dual CFT does not admit a local structure splitting of Hilbert Space of dual CFT, so in some sense assumption of locality of CFT, Black Hole Complementarity[4] and splitting of Hilbert Space of dual CFT in some pseudoprecise way are inconsistent eventually leading us to suggest that Black Hole spacetime does not admit a smooth Reimannian Goemetry or the principle of Causality of spacetime needs to be modified in a dramatic way.[5] Our

[1]We will be addressing the idea of notion of a manifold properly later in the paper.
[2]More primitively ideas from analysis like limits, differentiability and from differential geometry being applicable
[3]We will address and explain what we mean by diffeomorphisms in later section/sections.
[4]Which can be a possible alterntaive to resolution of diffeomorphism symmetry violation paradox
[5]We will address various aspects of it later in the paper

aim in the next section will be to address the mathematics of Reimannian geometry to address the paradox.

### 1.1 The tools necessary

As suggested in the introduction Professor Albert Einstein first introduced his Special Theory of relativity with two basic principles -**Einstein Equivalence Principle** and **universality of speed of Light or Causality** in his General Theory of Relativity he introduced the idea of Curvature which naturally adapts the machinery of Reimannian Geometry. Our aim in this section will be to understand Reimannian Geometry and ideas related to it so that we can make meaningful sense of assigning the data on a manifold locally and using these ideas to smoothly extend the mathematical data or the data on the topology of the manifold. The necessary tools will be as we will see are-
1)Topology, Topological Space
2) Manifold
3)Reimannian Geometry and Differential structure of Manifolds
4) **More about Differentiable Manifolds-** Charts and atlases, Tangent Space,Lie Derivatives, Lie Derivatives as the idea of Curvature on Manifolds, Vector Bundles, Embeddings of a manifold, Fibres and Foliations, Metric spaces, Metric space Structure on Manifolds, Extending local data on Manifolds via Lie Derivative a.k.a Affine Connection, Diffeomorphism symmetry of Manifold and its non trivial constraints on the structure of manifold.

#### 1.1.1 Motivation for Topology

Topology as a branch of mathematics grew from the work of Euler who tried studying the famous Bridge problem where it was asked to him if it was possible to cross five bridges arranged in a certain way such that no bridge is crossed more than once and also crossing all the bridges. AS we can observe this problem does not care about the length of bridges or the inclination of bridges but only in the way they are connected. This is the primary motivation of studying Topology that is studying how geometrical objects or shapes are linked to one another , so we see that to study these geometric properties we do not need to know about metric structure and also the Topology of a surface is a global property rather than a local property that is it depends on the complete nature of the surface.In the set theoretic sense topology as we will see in the next paragraph Topology tells how the elements of set are related to one another.A given set can have many topologies. While studying Point set Topology we will discuss some of the important topologies like Discrete Topology, Indiscrete Topology, Cofinite Topology...Using Point Set Topology as the base we will try discussing various other kinds of Topology defined on Manifolds like Differential Topology, Metric Topology, Geometric Topology, Algeabraic Topology briefly and as per the relevance needed in the paper...

#### 1.1.2 Topology, Topological Space

Topology by its name means study of shapes, initially introduced by Euler this branch of mathematics tries to study propoerties of objects that are independent of geometric notions like length, angle so in some sense it can be said for a smooth surface like a rubber ball

Topology tries to study the properties of the ball that remain invariant under deformations of the ball as long as we can retain the ball after deformation. Naturally the precise operations necessary are squashing of shape of ball generating a family of smooth surfaces that can be retraced back to ball. So we see that for a smooth surface[6] the topology studies an equivalence class of surfaces that have a non vanishing continuous rth order derivative everywhere defined on the surface.[7]Mathematicians thought to generalise the notion of topology as a notion described through set, which is preserved under set operation of finite intersections and arbitrary unions.A set together with its Topology(which is not unique ) describes a Topological Space.

### 1.1.3 Point set Topology

Given a set X with P(X) denoted as power set of X a topology $\tau(X)$ is defined as the subset of P(X) with collection of elements say $C_1, C_2, C_3, ...$ empty set $\phi$ and universal set X such that the set operations of arbitrary unions and finite intersection of elements of $\tau$ also belong to topology $\tau$.
So for arbitrary set of elements $C_i, C_j \in \tau$
$C_i \cap C_j \tau$ and $C_i \cup C_j \in \tau$ and
$\phi \in \tau$ , $X \in \tau$
Please note that we can have arbitrary number of intersections and arbitrary unions we have assumed 2 for just denoting the operation. Also note that keeping aside the notion of compactness of sets the number of elements in the open set can be infinite.We will worry about the cardinality[8] of sets later whenever necessary. Also there can be many possible Topologies for a given set as we will illustrate later in the subsection.

**Examples**
1)$x = \{a, b, c\}$ and $\tau = \{\phi, x, a, b\}$
We can see that given set x, $\tau$ is not a topology on x because the union of elements a and b does not belong to $\tau$.
2) $x = \{a, b, c\}$ and $\tau = \{\phi, x, \{a, b\}, \{b, c\}\}$ We again see that $\tau$ is not a Topology on x as intersection of elements $\{a, b\} \cap \{b, c\} = b$ which is not in $\tau$.
3) $x = \{a, b\}$ and $\tau = \{\phi, a\}$ Noticing carefully we see that set x is not contained in it so again it is not a topology on set x.
4) $x = \{a, b\}$ and $\tau = \{\phi, x, a\}$ We can see that all unions and arbitrary intersections are contained in the topology $\tau$ so set $\tau$ is one of the possible topologies of set x.
1)**Indiscrete Topology** Suppose $\tau = \{\phi, X\}$ then certainly it satisfies being a Topology. This is called Indiscrete or Trivial Topology.**We will discuss later why point set topology is always described between open sets.**

[6]A surface is said to be smooth if it has a smooth tangent space structure everywhere this is because to obtain a non smooth structure the first order or nth order derivative locally defined at some point on the manifold must be discontinuous

[7]These properties come under the section of study of Differential Topology which for now will not be our concern.

[8]Cardinality is number of elements in set

2)**Discrete Topology** Suppose $\tau = P(X)$ then certainly again $\tau$ satisifes being a Topology. This is called Discrete Topology. The 2 examples that we have discussed here are examples of point set topology which means the Topology $\tau$ contains finite number of elements. We will now discuss some other examples of Point Set Topology.

**Non trivial example of point set Topology**
3)**Cofinite Topology** Given a universal set X and a set A we define Cofinite Topology as the Topology $\tau$ on a set $A' = X - A$ such that the complement of set A $A'$ contains finite number of elements. Please note that sets X and A maybe finite or infinite however the complement of set A, $A'$ contains finite number of elements. We will verify that set $A'$ is a topology and is called Cofinite Topology.

**Larger and smaller Topology** Given a set x there can be many possible Topologies on the set x. Suppose $\tau_1, \tau_2$ are 2 such Topologies and $\tau_1 \subset \tau_2$ then the topology $\tau_2$ is larger topology and $\tau_1$ is smaller topology. **If the Topological space is also a manifold it might be plausible to make a comparison between 2 set of manifolds based on their dimensionality or cardinality.If they are contained in some regions, then it may or not be necessarily true that manifold with greater degrees of freedom is the larger topology, always. As an example suppose we consider a gravitating region which is assumed to be an isolated manifold then the shrinking of region with time increases the entropy globally though in some local region the entropy might decrease.**

## 2 Riemannian Geometry

Riemann generalised the notion of curved spaces famously known by his name.These are generally assumed to be curved spaces with a notion called curvature which might have been introduced earlier than Gauss but Carl Friedrich Gauss was probably the first person to give a systematic mathematical treatment of the notion of curvature in 2 dimensions. Sir Bernard Reimann generalised this notion to arbitrary dimensional curved space. An appropritate quantity to capture curvature is the commutator of Lie Derivative which gives a 4th order tensor called Reimann Tensor. Our aim will be to study these notions in a mre abstract way.The first abstract notion of curved space is a **manifold**.

### 2.1 Manifold

Mathematicians are good in abstracting things so that given some idea they can use the logic behind it so that the abstraction as the feature can be generalised to other or new relevant situations. We all are aware of surfaces, it is some region where we observe the pheonomenons in nature. Coloquially Manifold is an abstract generalisation of surfaces , it is a collection of curved surfaces glued together smoothly, the notion of being glued smoothly means that if we magnify over some little region of a manifold then it looks like a Euclidean Space , however our intent in defining a manifold is that it is a smoothly glued surface though it may not necessarily be everywhere Euclidean but the notion of smoothness is assumed to be essentially true.**A manifold can be assumed to be some kind of space that encodes geometric information[9].** The notion of manifold is taken from geometers exploration of world where he magnifies a region of earth with his maps as charts and the manifold is built together by gluing his charts which is called an atlas. The intent is that by smoothly gluing all the charts we can study the property of the surface which is assumed to be a manifold. Our aim in these sections will be to build the mathematical machinery required to study smooth manifolds. Motivated by the tools from this machinery we will then try to study generic properties of manifolds which may not necessarily be everywhere smooth.

#### 2.1.1 Charts and Atlas

We provide definitions of Charts , atlas and maps between them. Given a manifold M with coordinate charts $U_\alpha$ we deine a map $\phi_\alpha$ as -

$$\phi_\alpha : U_\alpha|_M \to R^n$$

where n is dimensionality of the Manifold M.In these section we will assume that manifold is smooth and differentiable everywhere to build the basics.The maps $\phi_\alpha$ are assumed to be smooth and differentiable and $U_\alpha|_M$ are called the coordinate charts and the union of all such charts $A = \cup_\alpha U_\alpha$ is called an atlas. A maximal atlas is an atlas with the least number of charts $U_\alpha$ such that their union covers the manifold M. We often hear that Spacetime is a differentiable manifold. Our motive through this section will be to give a meaningful definition of manifold , what is Topology , what are smooth Manifold, what is a topological manifold(if possible).These ideas will be discussed in great detail in this section. This section will be used to define the the notions used for describing a smooth differentiable manifold and its various properties. The subsection will be used to discuss loosely the paradoxes in framing spacetime as a manifold.

### 2.2 Diffeomorphism

Given that Manifold is a suitable generalisation of Curved space the fist thing we may desire is to be able to do calculus on Manifolds.As charts is a map from an open set in the manifold to $R^n$ the map should be assumed to be a smooth function.Also there can be many possible charts mapped to $R^n$ we want that if two coordinate charts intersect the map should be a homemorphism.This is suggested in figure 1.Homeomorphism between the coordinate charts means that the maps $\phi_1 \circ (\phi_2)^{-1}$ and $\phi_2 \circ (\phi_1)^{-1}$ are smooth maps from a subset of $R^n$to subset $R^n$. Compatible charts are also written as $C^\infty$ charts.

## 3 Diffeomorphism and Einstein principle of Equivalence

Professor Albert Einstein first introduced his Special Theory of relativity with two basic principles -Einstein Equivalence Principle and universality of speed of Light or Causality

[9]The geometric information is assumed to be something ill defined here it might not be some space that allows smooth differentiable structure always or it might be some abstract algeabraic structure that encodes some property of the set or elsewise geometric information can be encoded via some kind of morphism between abstract categorical objects

in his General Theory of Relativity he introduced the idea of Curvature which naturally adapts the machinery of Riemannian Geometry.Our aim in this section will be to generalise Special Theory of relativity in curved space using the formulation of Diffeomorphism symmetry group.However we will not rely on mathematical machinery of metric tensor for transformations which appears to be a bit misleading. Our intution is that there might not be a linear one to one map between distinct points in a Riemannian or Pseudo Riemannian manifold(Lorentzian in our example).This maybe because the manifold might get stretched significantly along angular directions so that there is not a nice one one map everywhere between distinct coordinates or in other words the spherical symmetry of the manifold might not always be present. However assuming the spacetime is 4 dimensional we assume there exists diffeomorphisms D between distinct regions of spacetime manifold or between 2 observers. We do not have a reliable indictaor of strong or weak gravity regime, one such notion can be the Ricci Scalar or also called Ricci Curvature.The manifold data in terms of metric for Schwarschild Spacetime is eventually written in terms of parameter M ,radial coordinate r, angular coordinates $\theta$ and $\phi$.For a generic manifold arbitrarily curved we do not expect radial symmetry but our expectation is that the spacetime manifold in some coordinate region can be assigned via some 4 vector t, r, $\theta$, $\phi$ .As the Lorentzian symmetry is described by Lorentz transformation in some suitable coordinate chart we assume the diffeomorphism symmetry group relates 2 different observers via Diffeomorphism symmetry group. However because we have not associated the tensorial nature of transformations or in other words the linear structure of transformations in strong curvature might not be always approximational.Suppose we use the diffeomorphism symmetry group in a suitable epsilon ball shaped region that contains some region of null hypersurface $\Sigma$ and a timelike region $\tau$.The introduction of affine connection in Riemannian geometry was to smoothly transport local data when the tensorial structure was not preserved we use similar intution to associate the 4×4 matrix and 2 vectors $\eta$(which is along null hypersurface) and a vector $\chi$ along timelike region, both $\eta$ and $\chi$ are 4 vectors.Our assumption is that like Black Hole spacetime a Manifold can be a union of such regions. Then $\eta$ and $\chi$ are related as-

$$\eta = D\chi \tag{3.1}$$

Since $\eta$ is along a null hypersurface its inner product with itself is zero. So $\eta^T\eta = 0$ also since $\chi$ is along a timelike region so $\chi^T\chi \neq 0$. Using equation 3.1 this implies

$$D^T D = 0 \tag{3.2}$$

Thus the Diffeomorphism symmetry matrix D relating timelike and null observers is singular so there does not exist diffeomorphisms between 2 such hypersurfaces.However this does not imply that there are no coordinate charts $\eta$ it is just that these coordinate charts are non invertible but they do exist.The hidden assumption of Causality does go in in this.

### 3.1 Reconstruction of Diffeomorphism Group D for 2 dimensional Minokowski spacetime

Our aim to provide equation 3.2 was to reconstruct Null Hypersurface for Black Hole Spacetime using this framework because tensorial transformations were inconsistent with

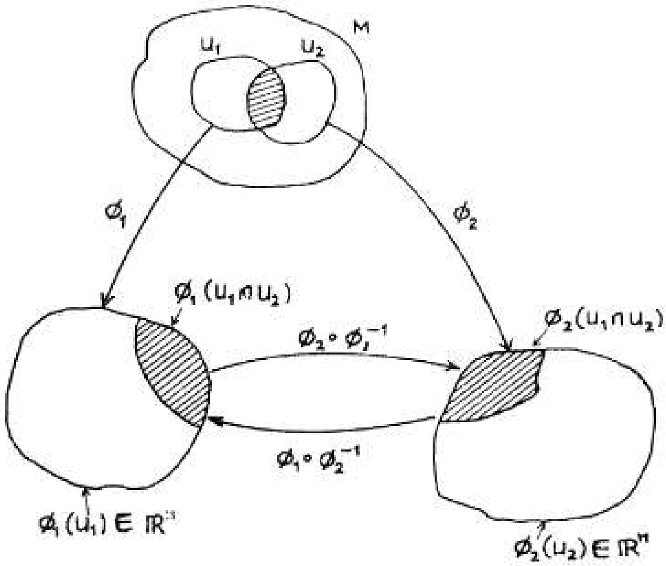


**Figure 1**. The compatibility condition for a smooth map is that given 2 coordinate charts with their intersecting regions the map in the intersecting region must be smooth and differentiable.

known modern versions of Black Hole Information Paradox like Professor Sameer Mathurs Information Pardox and AMPS version of information paradox.The example in previous section provides us an alternate way to reconstruct Null Hypersurfaces or event horizons.As of now we have not made any generic assumption so the Diffeomorphism symmetry group with 16 entries might be a complictaed function like D= D(t,r, $\theta$, $\phi$).One of the major challenges will be to reconstruct D for generic spacetimes ,already we have a result suggested in equation 3.2.We can now try to construct it for a timelike region in Minokowski spacetime for timelike region.Before considering a 4 vector let us assume a 2 dimensional Minokowski spacetime D= $\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}$ Suppose we consider 2 observers related via boosts and we label 2 vector observers as $\alpha$ and $\beta$ with $\beta$ as coordinates (t,r) so

$$\alpha_1 = d_{11}t + d_{12}r \tag{3.3}$$

and

$$\alpha_2 = d_{21}t + d_{22}r \tag{3.4}$$

Then

$$\alpha_1^2 - \alpha_2^2 = t^2 - r^2 \tag{3.5}$$

implies

$$d_{12}^2 = d_{21}^2 \tag{3.6}$$

Thus we do not necessarily assume that both of them are zero but they are either same or opposite. Also we find $d_{11}^2 = d_{22}^2$ this seems to be similar to 2 dimensional Lorentz transformation except that we did not use metric to scale the coordinates and we do find that the components of D are not necessarily symmetric.Here they are either of same or opposite sign.The connventional process of Tensor analysis in General Relativity assumes $d_{\mu\nu} = g_{\mu\nu}$ which does not seems to be right.Finding the matrix D (I am not calling it Diffeomorphism group because of equation 3.2) seems to be a non trivial process.

### 3.2 Diffeomorphism between timelike and null region in Minokowski spacetime

In this section our motivation will be to introduce Diffeomorphism between timelike and null region in Minokowski spacetime.Interestingly we will find that the elements of Diffeomorphism group satisfy Grassman Algebra.
Suppose we consider the diffeomorphism group between lightlike region in 2 dimensional spacetime with coordinates $t^{'}$ , $x^{'}$ and timelike region t,x then

$$\begin{bmatrix} t' \\ x' \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} t \\ x \end{bmatrix} \tag{3.7}$$

Using the condition as stated in equation 3.2 we find

$$d_{11} = id_{21} \tag{3.8}$$

$$d_{22} = -id_{12} \tag{3.9}$$

Also

$$(t')^2 - (x')^2 = 0 \tag{3.10}$$

Using 3.7, 3.8, 3.9 , 3.10 we can find one possible solution to the system of equation.Using first 3 equations We find

$$t' = id_{21}t + d_{12}x \tag{3.11}$$

$$x' = d_{21}t - id_{12}x \tag{3.12}$$

Equation 3.11 , 3.12 and 3.10 are consistent only when $d_{12}$ and $d_{21}$ are Grassman variables or matrices following Dirac Algebra because from the consistency condition we find

$$d_{12}^2 = d_{21}^2 = 0 \tag{3.13}$$

and

$$d_{12}d_{21} + d_{21}d_{12} = 0 \tag{3.14}$$

It seems like Diffeomorphism group in 2D Gravity is identical to Clifford Algebra.We do not know the implication of this result.Our motive will be to examine and possibly generalise this result for Non trivial cases when Gravity is dynamical or also called Pseudo Riemannian Geometry.Also from equation 3.11 and 3.12 we see that

$$t' = ix' \tag{3.15}$$

We find that Null diffeomorphism group D is

$$D = \begin{bmatrix} id_{21} & d_{12} \\ d_{21} & -id_{12} \end{bmatrix} \tag{3.16}$$

Since the time and spatial coordinate are complex valued we cannot necessarily say that velocity of such curves is imaginary.Our intution says that imaginary velocity of such curves is because of acausal diffeomorhisms. Though the usual rule of calculus as per equation 3.15 does tell us so.There is a Grassman variable alongside in equation 3.11 and 3.12 we do not know what is its implication or meaning.It seems surprising why and how Grassman variables appear in this situation.We will devote the next section to undertsnad Grassman numbers.Most of the things said here are taken from Wikipedia section on Grassman numbers.

### 3.3 Grassman numbers

Grassman numbers are square root of zero in the sense that they square to zero but may not necessarily be zero.They are anticommuting elements.The idea of anticommuting elements arises in lot of places in mathematics like in Differential geometry where differential forms are seen as anticommuting elements. Differential forms are normally defined in terms of derivatives on a manifold; however, one can contemplate the situation where one "forgets" or "ignores" the existence of any underlying manifold, and "forgets" or "ignores" that the forms were defined as derivatives, and instead, simply contemplate a situation where one has objects that anti-commute, and have no other pre-defined or presupposed properties. Such objects form an algebra, and specifically the Grassmann algebra or exterior algebra. The Grassmann numbers are elements of that algebra. The appellation of "number" is justified by the fact that they behave not unlike "ordinary" numbers: they can be added, multiplied and divided: they behave almost like a field.

**Matrix representation of Grassmanians** Elements $d_{21}$ and $d_{12}$ can be represented via matrices as-

$$d_{12} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad d_{21} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix} \tag{3.17}$$

This immediately raises the question what are these additional degrees of freedom. We say that they satisfy Anticommuting relation, hence they are some kind of Lie Algebra.We see that 4*4 matrices are the simplest representation of the elements of Null Diffeomorphism group which satisfies equation 3.13 and 3.14 in a non trivial way.**In general, a Grassmann algebra on n generators can be represented by $2^n \times 2^n$ square matrices.**

Physically, these matrices can be thought of as raising operators acting on a Hilbert space of n identical fermions,thus these elements might contain internal fermionic degrees of freedom and appear as matrix representation. Since the occupation number for each fermion is 0 or 1, there are $2^n$ possible basis states. Mathematically, these matrices can be interpreted as the linear operators corresponding to left exterior multiplication on the Grassmann algebra itself.Thus it seems that Fermionic degrees of freedom are preexistent in the Null Diffeomorphism group and to understand it we introduce the idea of a superspace.But before that we rewrite the null coordinates as stated in equation 3.11 and 3.12 using the Diffeomorphism group D and explore its implications.

$$t' = id_{21}t + d_{12}x$$

$$x' = d_{21}t - id_{12}x$$

This implies that $t$ and $x$ are themselves comprised of 4 elements and these 4 elements can be regarded as spacetime internal degrees of freedom and so is for $t'$ and $x'$.But the elements are such that they appear to be identity mostly but for non trivial scenarios where light cone stretches dramatically or at ultrarelativistic speeds they have other non trivial elements. We notice that equation 3.17 can be written using 2*2 Pauli matrices as-

$$d_{12} = \begin{bmatrix} \sigma_- & 0_{2*2} \\ 0_{2*2} & \sigma_- \end{bmatrix} d_{21} = \begin{bmatrix} 0_{2*2} & 0_{2*2} \\ \sigma_z & 0_{2*2} \end{bmatrix} \tag{3.18}$$

This immediately tells us that the null Diffeomorphism group is isomorphic to Algera of dirac matrices or they are 4*4 spinor elements.We also know the non trivial algebra of null diffeomorphism group as per equation 3.13 and 3.14.We assume t has 2 2*2 components $t_1$ and $t_2$ and x has components $x_1$ and $x_2$ and similarly t' has components $t'_1, t'_2$ and x' has components $x'_1, x'_2$.Then we find using equation 3.11 and 3.12

$$\begin{bmatrix} t'_1 \\ t'_2 \\ x'_1 \\ x'_2 \end{bmatrix} = \begin{bmatrix} i\sigma_- t_1 \\ \sigma_- t_2 + \sigma_z x_1 \\ \sigma_- t_1 \\ -i\sigma_z x_1 + \sigma_- t_2 \end{bmatrix} \tag{3.19}$$

Using equation 3.19 we see that

$$t'_1 = ix'_1 \tag{3.20}$$

but there is no direct such relation between $t'_2$ and $x'_2$.Their is a mixing of boosts in the second components of the elements of $t'$and $x'$.We also notice that if 4 vector elements $t$ and $x$ are real then the elements $t'$ and $x'$ are necessarily complex. Thus suggesting that such diffeomorphisms admit a complex Riemannian manifold structure. We also see that such elements hint towards spacetime elements having more than single degree of freedom but these degrees of freedom have real eigenvalues as long as there are no violations of Causality.Violation of causality implies splitting and mixing of real and complex degrees of freedom between spacetime elements.We see that 2 dimensional spacetime has many additional fermionic or internal degrees of freedom. **For a 4 dimensional spacetime**

**this seems to suggest $2^n \times 2^n$[10] or 256 dimensional degrees of freedom with 8 spacetime elements all together if also including complex degrees of freedom.** These additional degrees of freedom are in general complex valued.Finding the evolution equation for these degrees of freedom for a generic manifold appears to be a hard task.In flat spacetime which is an idelaisation to general spacetime these degrees of freedom do not interact with spacetime or bosonic degrees of freedom but in curved geometry or supergravity regime we do see such an interplay.It seems to be a plausible scenario for accounting for some part of Gravitational entropy in relativisitic scenarios where these degrees of freedom are different from identity and might be responsible for some part of entropy behind horizons.In general these degrees of freedom might be real or complex.Thus it seems that the natural avenue to study Spacetime manifold is to assume also these fermionic degrees of freedom and so as a complex Riemannian manifold.In certain scenarios spacetime might admit a specific structure called supersymmetry which is some intertwining of the bosonic and fermionic degrees of freedom but in general after breaking of supersymmetry fermionic degrees of freedom get hidden internally and do not appear visible to us at low speed but might cause significant changes to spacetime at relativisitic speeds.Thus the avenue to study Spacetime is to assume it to be graded or a superspace. Why these degrees of freedom might not be visible at small speeds(that is fermionic degrees of freedom appear identity) is a question of further investigation.

### 3.4 Null diffeomorhism Group for 1+ 1 d Minokowski spacetime

Using equation 3.18 and 3.16 we can find the null diffeomorphism group as-

$$D = \begin{bmatrix} 0 & 0 & \sigma_- & 0 \\ i\sigma_z & 0 & 0 & \sigma_- \\ 0 & 0 & -i\sigma_- & 0 \\ \sigma_z & 0 & 0 & -i\sigma_- \end{bmatrix} \tag{3.21}$$

We can verify that $D^T D = 0$ (as second coloumn is zero) and that D is a 64 element matrix but with 16 degrees of freedom as $\sigma_-, \sigma_z, i\sigma_-, i\sigma_z$.We also see that the diffeomorphism group matrix is not symmetric or hermitian.

Our motive now will be to understand generic properties of this manifold relying on its spinorial structure thus we proceed to understanding spinors and superspace which is natural way to understand spinors on a manifold.

### 3.5 Superspace

Our motivation here will be to study Superspace in the context of Minokowski spacetime.A superspace here is tensor product of Minokwoski spacetime with a set of Grassmann numbers at every point in spacetime.This is also called a Graded Minokowski spacetime and written as $R^{m,1|n}$ meaning R(m,1) is m+1 diemnsional Minokowski spacetime and n is the set of n grassmann numbers.We denote the space as $S(m+1|n) = R(m,1|n)$.Since the

[10]We have highlited this in section 3.3 on Grassmann numbers

square of Grassmann number is zero a generic point in superspace can be written as-

$$f(x,\theta) = f(x) + \Sigma_{A=1\to n}\theta^A f_A(x) + \Sigma_{A\neq BandA,B\in[1,n]}\theta^A\theta^B f_A(x)\wedge f_B(x)$$

The wedge product ensures that last term is antisymmetric in the indices A and B.

### 3.6 Null diffeomorphism group for 2+ 1 dimensional Minokowski spacetime

Given the superspace structure of 1+ 1 dimensional Minokowski spacetime we try investigating null diffeomorphism group for 2+1 dimensional Minokowski spacetime. For simplicity we label the diffeomorphism group as-

$$D = \begin{bmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{bmatrix}$$

As $D^T D = 0$ so the diagonal terms give

$$\begin{aligned} d_{11}^2 + d_{21}^2 + d_{31}^2 = 0 \\ d_{22}^2 + d_{12}^2 + d_{32}^2 = 0 \\ d_{13}^2 + d_{23}^2 + d_{33}^2 = 0 \end{aligned} \tag{3.22}$$

We assume

$$d_{21} = id_{11}cos(\theta_1), d_{31} = id_{11}sin(\theta_1) \tag{3.23}$$

$$d_{12} = id_{22}cos(\theta_2), d_{32} = id_{22}sin(\theta_2) \tag{3.24}$$

$$d_{13} = id_{33}cos(\theta_3), d_{23} = id_{33}sin(\theta_3) \tag{3.25}$$

The off diagonal terms give-

$$d_{11}d_{12} + d_{21}d_{22} + d_{31}d_{32} = 0 \tag{3.26}$$

$$d_{11}d_{13} + d_{21}d_{23} + d_{31}d_{33} = 0$$

$$d_{12}d_{13} + d_{22}d_{23} + d_{32}d_{33} = 0$$

We supply the ansatz in equations 3.41-3.43 into equations 3.44 this gives-

$$d_{11}d_{22}(i(cos\theta_2 + cos\theta_1) - sin\theta_1 sin\theta_2) = 0 \tag{3.27}$$

So

$$cos\theta_1 + cos\theta_2 = -isin\theta_1 sin\theta_2 \tag{3.28}$$

Similarly -

$$cos\theta_2 + cos\theta_3 = -isin\theta_2 sin\theta_3 \tag{3.29}$$

$$cos\theta_1 + cos\theta_3 = -isin\theta_1 sin\theta_3$$

If all the three parameters are equal then we find $cos\theta_1 = -i$ giving an imaginary boost again.Solving in general by elimination of these set of equations from 3.26-3.29 We find that the set of equations satisfy the quartic equation-

$$5z^4 + 2iz^3 - 6z^2 + 9 = 0$$

Here $z = e^{i\theta} = cos(\theta) + isin(\theta)$
The 3 set of $\theta$- $\theta_1$, $\theta_2$, $\theta_3$ along with cos($\theta$)=-i(when all 3 angles are equal) are the 4 sets of solution to the equation above. If all 3 angles are equal ,using these ansatz we find

$$D = \begin{bmatrix} d_{11} & id_{22}cos\theta_2 & id_{33}cos\theta_3 \\ id_{11}cos\theta_1 & d_{22} & id_{33}sin\theta_3 \\ id_{11}sin(\theta_1) & id_{22}sin\theta_2 & d_{33} \end{bmatrix} \tag{3.30}$$

We can rewrite this as-

If the three angles are equal then

$$D = \begin{bmatrix} d_{11} & d_{22} & d_{33} \\ -d_{11} & d_{22} & i\sqrt{2}d_{33} \\ i\sqrt{2}d_{11} & i\sqrt{2}d_{22} & d_{33} \end{bmatrix}$$

If We use this diffeomorphism group in a suitable coordinate chart between timelike and lightlike region we immediately arrive at a subtlety because $det(D) \neq 0$.Thus the angles relate with boost parameter in a non trivial way with diffeomorphism group such that det(D)=0 as the solution with same boosts is ruled out.This makes analyzing the algebra of the elements a non trivial task.As in equation 3.11 and 3.12 we rewrite the equation in matrix form as-

$$\begin{bmatrix} t' \\ x' \\ y' \end{bmatrix} = \begin{bmatrix} d_{11} & id_{22}cos\theta_2 & id_{33}cos\theta_3 \\ id_{11}cos\theta_1 & d_{22} & id_{33}sin\theta_3 \\ id_{11}sin(\theta_1) & id_{22}sin\theta_2 & d_{33} \end{bmatrix} \begin{bmatrix} t \\ x \\ y \end{bmatrix} \tag{3.31}$$

or

$$t' = d_{11}t + id_{22}cos(\theta_2)x + id_{33}cos(\theta_3)y \tag{3.32}$$

$$x' = id_{11}cos(\theta_1)t + d_{22}x + id_{33}sin(\theta_3)y \tag{3.33}$$

$$y' = id_{11}sin(\theta_1)t + id_{22}sin(\theta_2)x + d_{33}y \tag{3.34}$$

Also $t'^2 - x'^2 - y'^2 = 0$ and as the equation is true for any generic t,x,y,z implies following equations-

$$d_{11}^2 = d_{22}^2 = d_{33}^2 = 0 \tag{3.35}$$

$$d_{11}d_{22} + d_{22}d_{11} = 0 \tag{3.36}$$

$$d_{11}d_{33} + d_{33}d_{11} = 0 \tag{3.37}$$

$$d_{22}d_{33} + d_{33}d_{22} = 0 \tag{3.38}$$

This is identical to algebra we found in 2 Dimensions except we have 3 such pairs of Grassmann Variables satisfying the algebra as there are 3 pairs of elements. In 3+1 D

Minowkowski spacetime we can do the same computation and find that there are 6 set of anticommuting Grassman algebra elements.This algebra is the graded odd part of Lie Superalgebra with elements as Grassman numbers and it seems to be a generic property of elements of Minokowski Spacetime in any dimensions.As the algebra is graded odd part of Lie Superalgebra this might also be related to Supersymmetry and Fermionic degrees of freedom.

**Matrix representations**The simplest representation of 3 Generators of Grassman algebra has following Matrix representation-

$$d_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.39}$$

$$d_{22} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.40}$$

$$d_{33} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.41}$$

Thus the matrix D for 2+1 D Minowkowski spacetime contains 144 elements.

### 3.6.1 Implications of the Algebra for Causality

The[11] algebra of operators we described has an interesting connection with causality in spacetime physics.

**Propogation of Signals** : In causal systems, information or disturbances propagate in specific directions constrained by the light cone. Nilpotency can represent transitions that are limited in causal extent to one way or non reversibility.In causal structures, certain processes (like crossing a black hole event horizon) are irreversible. Nilpotent operators, which "annihilate" themselves when squared, could symbolize the one-way nature of such transitions.

**Anticommutation and Grassman structure:** The anticommutation relations reflect the fermionic behavior commonly seen in quantum field theory and supersymmetry. These structures are significant for causality in several ways:

**Spacetime Degrees of Freedom**: Grassmann numbers often represent fermionic degrees of freedom implying spacetime might have structure of superspace or supergeometry. Fermionic degrees of freedom might play an essential role when investigating relativisitic speed and might also be responsible for hiding inherent degrees of freedom. These fermionic coordinates might play a role in maintaining causal consistency when combining bosonic and fermionic sectors in theories of quantum gravity or supersymmetry.

[11] We thank Chatgpt for providing us with these remarkable insights.This section is entirely the work of Chatgpt and we are deeply thankful to it

### 3.7 Reconstruction of Diffeomorphism group for Null Hypersurface in Schwarschild Spacetime

Rather than adopting the conventional route of analytically extending the metric functional we assume the 4 vector along null hypersurface as $\eta$.We will use only one information that $\eta^T\eta = 0$ we will not assume any other information about this null hypersurface but for a 4 vector $\chi$ along timelike region owing to consistencies from experiments we assume the coordinate chart using $\chi$ in timelike region follows Raychaudhari equations and geodesic equations. Raychaudhari equation in a generic 4 dimensional spacetime is-

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma^2 + \omega^2 - R_{\mu\nu}\xi^\mu\xi^\nu + \xi^\alpha;\alpha \tag{3.42}$$

One might question if Einstein equations are valid near Black Hole Horizon given that the Diffeomorphism symmetry is absent.It was suggested in the paper[1] that Raychudhari equation can be derived independently as a Geometrical flow equation.This will be our motivation in presuming that Raychudhari equation are more fundamental and are valid even when there is an absence of Diffeomorphism symmetry in Black Hole Spacetime.We assume validity of Raychaudhari equation along Timelike region $\tau$ and along Null Hypersurface $\Sigma$ and that the two regions are related by equation 3.1.

However we do not have explicit form for Shear in the same paper[1] it is suggested that the shear potential in its equation (6) has an explicit dependence on volume of cross sectional surface.Our motive is to derive the explicit form of coordinate chart along null hypersurface using Raychaudhari equation along Timelike and Null hypersurface and relate the two via equation 3.1 and using geodesic equation for timelike region for the 4 vectors we try deriving the coordinate chart for null hypersuurface. We have an extra condition of zero norm of null hypersurfacebut we do not presume geodesic equation for null Hypersurface coordinates.In most scenarios Raychaudhari equation is derived using derivative of velocity vector field as in equation (1) of [2] .We do not know how to make an equivalence between Raychudhari equation using Geometrical flow and deriving it using the approach of [2].As suggested in the same paper after equation (14) Raychudhari equations are identities and not equations and therefore are valid even when Einstein equations owing to absence of diffeomorphism symmetry might themselves may not necessarily be valid.For notational simplicity we rewrite these identities suggested in the paper.

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma^2 + \omega^2 - R_{\mu\nu}\xi^\mu\xi^\nu + \xi^\alpha;\alpha \tag{3.43}$$

$$\frac{d\sigma_{ab}}{d\tau} = -\frac{2}{3}\theta\sigma_{ab} - \sigma_{ac}\sigma^c_b - \omega_{ac}\omega^c_b + \frac{1}{3}h_{ab}(\sigma^2 - \omega^2) + C_{cbad}v^c v^d + \frac{\tilde{R}_{ab}}{2} \tag{3.44}$$

$$\frac{d\omega_{ab}}{d\tau} = -\frac{2}{3}\theta\omega_{ab} - 2\sigma^c_{[b}\omega_{a]c} \tag{3.45}$$

Where $\sigma^2 = \sigma_{ab}\sigma^{ab}$($\sigma_{ab}$ has Symmetric indices) $\omega^2 = \omega_{ab}\omega^{ab}$ ($\omega_{ab}$) has antisymmetric indices.

$C_{abcd}$ is the weyl tensor and

$$\tilde{R}_{ab} = h_{ac}h_{bd}R^{cd} - \frac{1}{3}h_{ab}h_{cd}R^{cd} \tag{3.46}$$

Here $h_{ac}$ is the projection tensor. A projection tensor is a tensor orthogonal to 4 velocity vector.So it is defined as-

$$h_{ac} = g_{ac} - u_a u_c \tag{3.47}$$

where $u_a$ is a timelike velocity vector and is normalised to 1.One may question how to find the timelike velocity vector we assume these can be solved using set of geodesic equations which are solution to Einstein equations.The 4 set of coupled differential equations are-

$$\frac{d^2t}{d\tau^2} + \frac{\alpha}{\rho(\rho-\alpha)}\frac{dt}{d\tau}\frac{d\rho}{d\tau} = 0 \tag{3.48}$$

$$\rho^{\cdot\cdot} + \frac{\alpha(\rho-\alpha)}{2\rho^3}(t^{\cdot})^2 - \frac{\alpha}{2\rho(\rho-\alpha)}(\rho^{\cdot})^2 - (\rho-\alpha)(\theta^{\cdot})^2 - (\rho-\alpha)sin^2(\theta)(\phi^{\cdot})^2 = 0 \tag{3.49}$$

$$\theta^{\cdot\cdot} + \frac{2}{\rho}\rho^{\cdot}\theta^{\cdot} - sin\theta cos\theta(\phi^{\cdot})^2 = 0 \tag{3.50}$$

$$\phi^{\cdot\cdot} + \frac{2}{\rho}\rho^{\cdot}\phi^{\cdot} + 2cot\theta\phi^{\cdot}\theta^{\cdot} = 0 \tag{3.51}$$

$u_a = \frac{dx_a}{d\tau}$ where $x_a$ is one of the four components of the 4 set of geodesic equations.Interestingly we asked chatgpt to find solution to these set of equations and it gave us following solution-

$$u_t = \frac{E}{1-2M/r} \tag{3.52}$$

$$u_r^2 + (1-2M/r)(E^2 - (1-2M/r)u_\theta^2 - r^2u_\phi^2) = 0 \tag{3.53}$$

$$u_\theta = \frac{l_1}{r^2} \tag{3.54}$$

$$u_\phi = \frac{l_2}{r^2 sin^2\theta} \tag{3.55}$$

Similarly for Null Hypersurface the authors in [2] suggest in equation (17) the Raychaudhary equation-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \sigma'^2 - \omega'^2 = -R_{ab}k^ak^b \tag{3.56}$$

Please note in equation 3.56 right hand side carries only symmetric terms in its indices so even if we assume metric to be asymmetric it is only the symmetric part that contributes in right hand side.This simplifies a lot of work for solving the equation in terms of symmetric indices.Also we will see that we need the equation for null hypersurface in 2 dimension so Ricci tensor simplifies significantly. Please note here the primed quantitites are the expansion, rotation and shear for the null geodesic congruence.Like equation (3.24) and (3.25) we have analagous equation for $\sigma'$ and $\omega'$ as tensorial equations. These are-

$$\frac{d\sigma'_{ab}}{d\lambda} = -\theta\sigma'_{ab} + C_{cbad}v^cv^d \tag{3.57}$$

$$\frac{d\omega'_{ab}}{d\lambda} = -\theta\omega_{ab} \tag{3.58}$$

### 3.8 Interpretation of Expansion

In section 2.4.8 of book-A relativist toolkit by Prof. Eric Poisson it is mentioned that $\theta$ is fractional rate of change of area of hypersurface.We assume the argument so

$$\theta' = \frac{1}{A}\frac{dA}{d\lambda} \tag{3.59}$$

Here A is the area of Null Hypersurface. Similarly

$$\theta = \frac{1}{V}\frac{dV}{d\tau} \tag{3.60}$$

We do not yet know a way to find Area A and Volume V however suppose we consider foliation of a cauchy slice just at the instant a balck hole is formed we know using the langauge of differential forms that

$$V = \int_{\tau} \sqrt{(detg)} d\chi_1 \wedge d\chi_2 \wedge d\chi_3 \wedge d\chi_4 \tag{3.61}$$

Assuming foliation is along one of the coordinates say $\chi_4$ we have

$$V = \int_{\tau} \sqrt{(detg)} d\chi_1 \wedge d\chi_2 \wedge d\chi_3$$

Here the integartion needs to be done over timelikeregion $\tau$ along the black hole spacetime outside the null hypersurface. Similarly in that slice we have

$$A = \int_{\Sigma} \sqrt{(deth)} d\eta_1 \wedge d\eta_2 \wedge d\eta_3 \tag{3.62}$$

Again we have to integrate the null hypersurface region $\Sigma$ along with the condition $\eta^T \eta = 0$. Assuming such a foliation can be done so as to eliminate one of the coordinates of $\eta$ we have 2 parameters for integration of A. So the Area is a 2 dimensional hypersurface which makes the analysis of the induced metric in 2 dimensions much simpler.But we will not assume that null hypersurface metric is diagonal or symmetric in its indices hence there are 4 components for the metric.Also because spacetime outside black hole horizon satisifies vacuum Einstein equation so we have $R_{\mu\nu} = 0$ for such a Cauchy slice. However we notice immediately that we have presumed an infinitesimal ball shaped region hugging both timelike region $\tau$ and null hypersurface region $\Sigma$.This is presumed to happen along some Cauchy slice containing both the regions but as we start foliating such Cauchy slices we will find that there is a blueshift in future so the blueshift and greybody factors will back react at late times distorting our construction to recosntruct null hypersurface.It is also questionable if such diffeomprphisms even exist or not as in section 3.4 and in equation 3.21 we found that the diffeomprphism group could have been zero also.In that scenario we assumed it to be non zero as per equation 3.21 and this lead us to the notion of superspace. We also need to know the induced metric determinants g and h.Hopefully we know the metric along timelike region $\tau$ as Schwarschild metric owing to experimental observations however we can not say so for null region $\sigma$. So we can proceed to link the set of equations derived in section 3.7

of the paper with coordinate charts along timelike region and null hypersurface and then link them using equation 3.1.This will help us to find Null hypersurface modulo one more condition/set of conditions.We assume the condition is smooth variation of coordinates as we move along a neighbhouring region from timelike surface to null hypersurface.This can be stated as the condition that timelike 4 vector motivated in equation 3.1 $\chi$ as displaced along a neighbhouring region along null hypersurface $\Sigma$ is same as $\eta$.Thus

$$\chi|_{\Sigma} = \eta \tag{3.63}$$

Or it means that as the radius of diffeomorphism group approaches zero the matrix D in equation 3.1 becomes identical to identity.For small deformations we assume the diffeomorphism group can be expanded via its set of generators close to identity.This will provide us the algebra of the diffeomorphism symmetry group. Hopefully the set of equations from 3.43 to 3.62 can be assumed to be sufficiently able to develop the coordinate chart for smoothly reconstructing the null hypersurface of Black Hole spacetime. **Until now we have only supplied geometrical identities and not an equation, we attempt to provide a consistently complete set of solutions by providing few more set of equations known as Gauss-Codazzi equation and Israel Junction condition as suggested in Chapter 3 of Eric Poisson which ensures smoothness of Horizon but it seems like we do not require those whether they are required or not is for now a question of further careful examination.These equations can be taken directly from Eric Poisson textbook but we will comment about it later**.
Because a null hypersurface has zero norm a manifold with an oriented normal along timelike region needs to change its sign necessarily.In the langauge of fibre bundles this means that if fibres of some base manifold which admits a suitable foliation necessarily travels back in time as it passes from timelike hypersurface to null hypersurface.But proving this statement rigorously requires deep theorems and results from Riemannian geometry and Differential geometry but certainly this process does not seem to indicate more subtle gravitational effects like Topology change which is captured using De Rham Cohomology.However just to make some progress we now attempt to solve set of equations from 3.43 to 3.62 using an assumption in equation 3.61 and 3.62 that the change in expansion along timelike region and null hypersurface is carried only as a function of metric so that while taking derivative of volume V and area A the change is carried only as function of metric.In particular we use these 2 more sets of equations that simplify 3.59 and 3.60 to -

$$\theta' = \frac{1}{\sqrt{h}}(\frac{d(\sqrt{deth})}{d\lambda}) \tag{3.64}$$

$$\theta = \frac{1}{\sqrt{g}}(\frac{det\sqrt{g}}{d\tau}) \tag{3.65}$$

For simplicity we write the null metric without indices being symmetric in 2 dimensions as-

$$h_{\mu\nu} = \begin{bmatrix} h_{00} & h_{01} \\ h_{10} & h_{11} \end{bmatrix} \tag{3.66}$$

Similarly the inverse metric has the form-

$$h^{\mu\nu} = \frac{1}{deth}\begin{bmatrix} h_{11} & -h_{01} \\ -h_{10} & h_{00} \end{bmatrix} \tag{3.67}$$

where $deth = h_{00}h_{11} - h_{01}h_{10}$
In 2 dimensions we find

$$\theta' = \frac{1}{\sqrt{deth}}\frac{d(\sqrt{deth})}{d\lambda}$$

So

$$\theta' = \frac{1}{2(h_{00}h_{11} - h_{01}h_{10})}\frac{d(deth)}{d\lambda} = \frac{1}{2deth}\frac{d(deth)}{d\lambda} = \frac{1}{2}\frac{d}{d\lambda}ln(deth) \tag{3.68}$$

Also in 2 dimensions in presence of asymmetry of metric or torsion we have the expression only for symmetric indices here owing to symmetry of right hand side of equation 3.56.Also in 2d the expression simplifies so -

$$R_{\mu\nu} = \frac{R}{2}h_{\mu\nu} \tag{3.69}$$

Asymmetric metric implies an asymmetric line element which indicates that if tensor analysis for line element is valid then the event horizon has non commutative nature though we do not claim it here because we do not assume that metric always is a fundamental arena for taking inner product for computing line element. It is only in Minokowski spacetime we presume it is true.Also in equation 3.56 $k^a = \frac{dx^a}{d\lambda}$ where $x^a$ are coordinates on null hypersurface.Thus equation 3.56 simplifies to-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \sigma'^2 - \omega'^2 = -(R/2)h_{ab}k^a k^b \tag{3.70}$$

where $\theta'$ is as suggested in equation 3.68.In 2 dimensions we have another simplification the weyl tensor vanishes so equation 3.58 becomes-

$$\frac{d\sigma'_{ab}}{d\lambda} = -\theta\sigma'_{ab} \tag{3.71}$$

$$\frac{d}{d\lambda}ln(\sigma'_{ab}) = -\theta$$

and

$$\frac{d\omega'_{ab}}{d\lambda} = -\theta\omega'_{ab} \tag{3.72}$$

To proceed further we use bit of tensor analysis.
We multiply equation 3.71 by $\sigma'^{ab}$ so we find-

$$\sigma'^{ab}\frac{d\sigma'_{ab}}{d\lambda} = -\theta\sigma'_{ab}\sigma'^{ab} = -\theta\sigma'^2 = \frac{d(\sigma'^{ab}\sigma'_{ab})}{d\lambda} - \sigma'_{ab}\frac{d\sigma'^{ab}}{d\lambda} \tag{3.73}$$

So

$$\sigma'_{ab}\frac{d\sigma'^{ab}}{d\lambda} = \frac{d\sigma'^2}{d\lambda} + \theta\sigma'^2$$

Multplying above equation by $\sigma'^{ab}$ we find-

This simplifies to-

$$\frac{d\sigma'^{ab}}{d\lambda} = \sigma'^{ab}(\theta + \frac{1}{\sigma'^2}\frac{d\sigma'^2}{d\lambda}) \tag{3.74}$$

Equation 3.74 can be written as-

$$\frac{d}{d\lambda}ln\sigma'^{ab} = (\theta + \frac{1}{\sigma'^2}\frac{d\sigma'^2}{d\lambda}) \tag{3.75}$$

From 3.75 and 3.73 we see triviality relation for $\sigma$.We can argue similar triviality relation for $\omega$ and using dimensional analysis in 3.70 we see that $\sigma'^2$ scales as $\frac{1}{\lambda^2}$ by the symmetry of equations for $\omega'$ with $\sigma'$ we see similar scaling for $\omega'^2$.Please note that we are not resorting to tensor analysis so the best we can find is triviality relations for $\sigma$ and $\omega$.We reduce equation 3.70 in the form-

$$\frac{d\theta'}{d\lambda} + \frac{\theta'^2}{2} + \frac{\alpha}{\lambda^2} = -(R/2)h_{ab}k^a k^b \tag{3.76}$$

Here $\alpha$ is determined by gluing conditions at the timelike and null hypersurface.Thus alpha eliminates equations for $\sigma$ and $\omega$.We substitute 3.68 into 3.76 and find the differential in terms of metric has the form-

$$\frac{1}{(2deth)}\frac{d^2(deth)}{d\lambda^2} - \frac{3}{8(deth)^2}(\frac{d(deth)}{d\lambda})^2 + \frac{\alpha}{\lambda^2} = -(R/2)h_{ab}k^a k^b \tag{3.77}$$

In 2 dimensions we can simplify the expression for Ricci scalar and we find it in terms of metric and its derivatives as-

$$R_\Sigma = -\frac{1}{deth}\frac{d^2deth}{d\lambda^2} + \frac{1}{2(deth)^2}(\frac{d(deth)}{d\lambda})^2 \tag{3.78}$$

Substituting for R from equation 3.78 into 3.77 we find the equation has the form-

$$\frac{1}{deth}(2 - \frac{1}{2}h_{ab}k^a k^b)\frac{d^2deth}{d\lambda^2} + \frac{1}{(deth)^2}(-\frac{3}{8} + \frac{1}{4}h_{ab}k^a k^b)(\frac{d(deth)}{d\lambda})^2 + \frac{\alpha}{\lambda^2} = 0 \tag{3.79}$$

### 3.8.1 Gauss Codazzi equation and Extrinsic curvature

We see that we need to find for $\alpha$ and induced metric h on null hypersurface .To proceed further we impose Gauss Codazzi equation[12] which relates Ricci scalar of 4 dimensional spacetime to Ricci scalar on its hypersurface $\Sigma$, $R_\Sigma$ via extrinsic curvature on hypersurface and trace of extrinsic curvature as-

$$R = R_\Sigma + K_{ab}K^{ab} - K^2 \tag{3.80}$$

In 4 D spacetime or any of its foliation[13] not enclosing null hypersurface we see or assume R=0. So

$$R_\Sigma + K_{ab}K^{ab} - K^2 = 0 \tag{3.81}$$

[12]This equation can be found in Sir Eric Poisson textbook, we searched it in chatgpt and we are deeply obliged to chatgpt for the same :)...

[13]Please note that Ricci scalar R is a geoemtrical property of Spacetime independent of how we foliate the spacetime so the equation R=0 is always valid.

### 3.8.2 Extrinsic curvature for null hupersuface

Extrinsic curvature mentioned in equation 3.80 is a measure of how much the hypersurface curves in the ambient spacetime.If the hypersurface is flat we expect extrinsic curvature to be zero.For null hypersurface thus it is a good measure to capture the geometry of null hypersurface which aligns with our aim to capture its geometrical properties.

**Formal definition** Given the induced metric $h_{ab}$ on our hypersurface we define the extrinsic curvature as-

$$K_{ab} = h^c_a h^d_b \nabla_c n_d \tag{3.82}$$

Thus it measures in some sense the component of derivative of normal vector along the direction of induced metric.Here $\nabla_c$ is the covariant derivative with reference to 4 d Spacetime.The equation has 4 dimensional covariant derivative because extrinsic curvature is a measure of how the normal vector changes in 4 dimensional ambient spacetime, please note that given a hypersurface we always have a normal vector along the gradient of hypersurface.A precise reasoning of the definition can be found in Eric Poisson textbook. Using equation 3.82 we proceed to with equation 3.81 of the paper.Please note that $K^{ab}$ is defined as-

$$K^{ab} = h^a_e h^b_f \nabla^e n^f \tag{3.83}$$

$$K = tr(K_{ab}) = tr(h^c_a h^d_b \nabla_c n_d) = h^c_a h^c_b \nabla_c n_c \tag{3.84}$$

Substituting equation 3.82, 3.83 and 3.84 into equation 3.81 we find the following equation-

$$R_\Sigma + h^c_a h^d_b \nabla_c n_d h^a_e h^b_f \nabla^e n^f - h^c_a h^c_b \nabla_c n_c h^d_a h^d_b \nabla_d n_d = 0 \tag{3.85}$$

For simplicity we write the equation in terms of lower components of metric so it has the form-

$$R_\Sigma + h_{ac} h_{bd} \nabla^c n^d h^{ae} h^{bf} \nabla_e n_f - h_{ac} h_{bc} \nabla^c n^c h^{da} h^{db} \nabla^d n^d = 0 \tag{3.86}$$

We use equation 3.67 which is-

$$h^{\mu\nu} = \frac{1}{deth} \begin{bmatrix} h_{11} & -h_{01} \\ -h_{10} & h_{00} \end{bmatrix}$$

We noticed that raising and lowering indices in terms of tensor is consistent with taking inverses but raising and lowering with reference to metric is not a justified procedure. To proceed further we use tensor contraction and find-

$$R_\Sigma + \nabla_c n_d \nabla^c n^d - (\nabla_c n^c)^2 = 0 \tag{3.87}$$

**Further simplification** Above equation can be rewritten as-

$$R_\Sigma = -(|\nabla n|^2) + (divn)^2 \tag{3.88}$$

Please note that $|\nabla n|$ is a tensor here and norm is taken by taking modulus of this tensor.This equation explicitly relates the intrinsic curvature of the hypersurface to the normal's gradient behavior in the ambient spacetime.

**Physical meaning** If $R_\Sigma = 0$ then the expansion and distortion of the hypersurface(measured by divergence of normal vector) exactly balance each other.
If the normal is geodesic i.e $\nabla_c n_d = 0$ then the hypersurface is flat which is analog of Local flatness theorem or locally Minkowski spacetime hypersurfaces.We thus have 2 equations to solve for null hypersurface equation 3.88 and equation 3.79.
**Computing normal vector**The hypersurface of Schwarschild spacetime can be assumed to be all region in exterior of event horizon. Assuming such a hypersurface has coordinates $\eta$ then the hypersurface is given by equation

$$\eta^T \eta = 0 \tag{3.89}$$

We parametrise such surface with coordinates $\eta_0, \eta_1, \eta_2, \eta_3$ then we get the null equation 3.89 as-

$$\eta_0^2 - \eta_1^2 - \eta_2^2 - \eta_3^2 = 0 \tag{3.90}$$

We can parametrise such surface as- $\eta = \eta_o(1, cos\theta cos\phi, cos\theta sin\phi, sin\theta)$.A possible foliation of such hypersurface can be constant $\eta_0$ slices. The gradient tensor or transformation matrix with respect to coordinates $\eta_0, \theta, \phi$ has the form-

$$T = \begin{bmatrix} 1 & cos\theta cos\phi & cos\theta sin\phi & sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & -\eta_0 sin\theta cos\phi & -\eta_0 sin\theta sin\phi & \eta_0 cos\theta \\ 0 & -\eta_0 cos\theta cos\phi & \eta_0 cos\theta cos\phi & 0 \end{bmatrix} \tag{3.91}$$

Please note that since there are three parametric coordinates so one of the row is entirely zero(2nd row). Then the normal vector is $n = T^{\leftrightarrow}.\eta$ so we find

$$n = \begin{bmatrix} 1 & cos\theta cos\phi & cos\theta sin\phi & sin\theta \\ 0 & 0 & 0 & 0 \\ 0 & -\eta_0 sin\theta cos\phi & -\eta_0 sin\theta sin\phi & \eta_0 cos\theta \\ 0 & -\eta_0 cos\theta cos\phi & \eta_0 cos\theta cos\phi & 0 \end{bmatrix} \begin{bmatrix} \eta_0 \\ \eta_0 cos\theta cos\phi \\ \eta_0 cos\theta sin\phi \\ \eta_0 sin\theta \end{bmatrix} \tag{3.92}$$

Please note that since the magnitude of $\eta = 0$ so we did not normalise the above equation.We find n as-

$$n = \begin{bmatrix} 2\eta_0 \\ 0 \\ 0 \\ \eta_0^2 cos^2\theta cos2\phi \end{bmatrix} \tag{3.93}$$

Using equation 3.93 we can compute norm of gradient of n and divergence of n using the transformation matrices.The transformation matrix or gradient matrix $T'$ for n is-

$$T'(n) = \begin{bmatrix} 2 & 0 & 0 & 2\eta_0 cos^2\theta cos2\phi \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2\eta_0^2 sin2\theta cos2\phi \\ 0 & 0 & 0 & -2\eta_0^2 cos^2\theta sin2\phi \end{bmatrix} \tag{3.94}$$

We know that $\nabla n = T'(n)$,hence $(\nabla n)^2 = Tr(T'(n)^2)$ Simplifying we find-

$$|\nabla n|^2 = 4 + 4\eta_0^2 cos^4\theta cos^4\phi + 4\eta_0^4(cos^4\theta sin^4\phi + sin^4\theta cos^4\phi) \tag{3.95}$$

Using 3.93 we find $div(n) = 2 - 2\eta_0^2 cos^2\theta sin2\phi$ So we find

$$div^2(n) = 4 + 4\eta_0^4 cos^4\theta sin^2 2\phi - 8\eta_0^2 cos^2\theta sin2\phi \tag{3.96}$$

Substituting 3.95 and 3.96 into 3.88 we find $R_\Sigma$ as-

$$R_\Sigma = -4\eta_0^2 cos^2\theta(cos^2\theta cos^4\phi + 2sin^2\phi) - 4\eta_0^4(cos^4\theta sin^4\phi + sin^4\theta cos^4\phi - cos^4\theta sin^2\phi) \tag{3.97}$$

Please note that in any coordinate chart there is no isometry in the Ricci Scalar along null hypersurface as expected from our intution and also in general the surface has negative Ricci scalar.Also along constant time slices we can assume $\eta_0$ to depend on Quasi local mass measured at the Black Hole horizon or at infinity if the manifold is non compact or at finite $\eta_0$ if the manifold is compact.It is still yet to be verified if the Manifold satisfies Grassmannian structure as eluded in Minkowski case.We also have an equation for $R_\Sigma$ as a differential equation for the metric in equation 3.78. Equating the two equations we find-

$$-\frac{1}{deth}\frac{d^2deth}{d\lambda^2} + \frac{1}{2(deth)^2}(\frac{d(deth)}{d\lambda})^2 = -4\eta_0^2 cos^2\theta(cos^2\theta cos^4\phi + 2sin^2\phi) \tag{3.98}$$

$$-4\eta_0^4(cos^4\theta sin^4\phi + sin^4\theta cos^4\phi - cos^4\theta sin^2\phi)$$

We also have equation 3.79 in our hand to solve for the metric.We rewrite that equation here for simplicity-

$$\frac{1}{deth}(2 - \frac{1}{2}h_{ab}k^ak^b)\frac{d^2deth}{d\lambda^2} + \frac{1}{(deth)^2}(-\frac{3}{8} + \frac{1}{4}h_{ab}k^ak^b)(\frac{d(deth)}{d\lambda})^2 + \frac{\alpha}{\lambda^2} = 0$$

Equation 3.79 can be simplified as it has a scaling symmetry we assume $deth = A\lambda^n$ then the left hand side of equation 3.98 has the form-

$$-n(n-1) + \frac{n^2}{2} = -n(\frac{n}{2} - 1) = -4\eta_0^2 cos^2\theta(cos^2\theta cos^4\phi + 2sin^2\phi) \tag{3.99}$$

$$-4\eta_0^4(cos^4\theta sin^4\phi + sin^4\theta cos^4\phi - cos^4\theta sin^2\phi)$$

Assuming given solution exists for det h we find that Ricci scalar is always negative if $n(n+2) > 0$.

These anasatz also simplify equation 3.79 as-

$$(2 - \frac{1}{2}h_{ab}k^ak^b)n(n-1) + (-\frac{3}{8} + \frac{1}{4}h_{ab}k^ak^b)n^2 + \alpha = 0 \tag{3.100}$$

This can be grouped as-

$$\frac{13}{8}n^2 - 2n - \frac{h_{ab}k^ak^b}{2}n(\frac{n}{2} + 1) + \alpha = 0 \tag{3.101}$$

Equation 3.99 and 3.101 can be useful to compute the metric and relevant quantities for null hypersurfaces.Parameter $\alpha$ is determined using gluing conditions for null hypersurface and timelike region.We still need to find the null Diffeomorphism matrix D.

### 3.9 Geometrical significance of Ricci scalar $R_\Sigma$

The Ricci scalar R encodes information about volume changes, focusing of geodesics, and curvature effects in a given spacetime or hypersurface. The sign of R has direct geometric and physical interpretations-
Ricci scalar tells ushow volume changes in curved space.Given an infinitesimal colume V, its rate of expansion is related to R as[14]-

$$\frac{d^2V}{d\tau^2} = -R_{\mu\nu}\xi^\mu\xi^\nu V + \frac{2}{3}\theta^2 V \tag{3.102}$$

For simplicty we assume the background is weakly curved so that second term in equation 3.102 is infinitesimal with reference to first term.Then we have following implications-
**Implications for Positive R**
If $R > 0$ then volume contracts as geodesics converge.Also manifolds with positive Ricci scalr tend to be compact.One example of such space is De-Sitter Space.
**Implications for negative R**
If $R < 0$ volume expands as geodesic diverge.One such example of such a space is Black Hole Interior[15] and Ads- Space.The Causal structure of spacetime is always a Hyperbolic manifold in curved spaces.

#### 3.9.1 Derivation of Volume Change equation using Raychaudhari equation

We restate Raychaudhari equation in 4 dimensional spacetime here-

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma^2 + \omega^2 - R_{\mu\nu}\xi^\mu\xi^\nu$$

We note

$$\frac{dV}{d\tau} = \theta V \tag{3.103}$$

Taking another derivative and substituting equation 3.103 we find-

$$\frac{d^2V}{d\tau^2} = V\frac{d\theta}{d\tau} + \theta^2 V \tag{3.104}$$

To proceed further we assume irrotatiional and shear free spacetime so $\sigma = \omega = 0$[16] With this assumption Raychaudhari equation takes the form-

$$\frac{d^2V}{d\tau^2} = -R_{\mu\nu}\xi^\mu\xi^\nu V + \frac{2}{3}\theta^2 V \tag{3.105}$$

Using 3.103 this simplifies to-

$$\frac{d^2V}{d\tau^2} = -R_{\mu\nu}\xi^\mu\xi^\nu V + \frac{2}{3V}(\frac{dV}{d\tau})^2 \tag{3.106}$$

[14] We will derive this equation in next section using Raychaudhari equations

[15] This is because the interior of Black Hole always grows in size and as per equation 3.102 this can only happen when R is less than zero

[16] We saw that these terms add a scaling constant to the equation earlier so they might change the equation only by constant terms, however given appropriate boundary conditions we can still solve the equations with the terms.

### 3.10 Solving for timelike region

To find the Diffeomorphism matrix D we need to solve for timelike region.To solve for timelike region for the coordinates we need to solve for set of equations from 3.43 to 3.61 and provide the smoothness condition as suggested in equation 3.63. However the crux of all is in equations 3.52-3.55 which needs to be integrated and solved to find explicit dependence of coordinates in terms of parameter $\tau$.

**Proceeding with solution** To simplify we notice that equations can be solved via division of parameter tau in equation for $u_r$ and $u_\theta$.We find that we can solve for equations in terms of dr d$\theta$ .Using 3.53 and 3.54 and substituing for $u_\theta$ and $u_\phi$ from 3.54 and 3.55 we find-

$$\frac{dr}{d\theta} = \frac{r^2}{l_1}\sqrt{(1-2M/r)((1-2m/r)\frac{l_1^2}{r^4} + \frac{l_2^2}{r^2 sin^2\theta} - E^2)} \tag{3.107}$$

This is differential equation for $r(\theta)$ which can be solved using ellitpic[17] integrals.To solve the above equation we introduce a variable u=$\frac{1}{r}$ so that $\frac{dr}{d\theta} = \frac{-1}{u^2}\frac{du}{d\theta}$.
Rewriting equation in terms of u we find-

$$\frac{du}{d\theta} = -\frac{u^2}{l_1}\sqrt{(1-2Mu)((1-2Mu)l_1^2 u^4 + \frac{l_2^2 u^2}{sin^2\theta)} - E^2)} \tag{3.108}$$

**Change of variables for ellitpic integrals form**We define a new variable $z = \frac{u}{u_{max}}$ where $u_{max}$ is turning point of orbit determined from effective potential conditions.However this procedure though computable is too cumbersome, so we make an approximation.Because we need to find solution near r=2m we assume a pertuabtion parameter $r' = r - 2m$ and expand perturbatively these set of equations.We asked elegant chatgpt to compute the equations and it gave us the following procedure by approximating radial equation pertuabtively.The radial equation is-

$$(\frac{dr}{d\tau})^2 + (1-2M/r)(E^2 - (1-2M/r)\frac{l_1^2}{r^4} - \frac{l_2^2}{r^2 sin^2\theta}) = 0 \tag{3.109}$$

Substituting $r' = r - 2M$ we find the differential equation has the form $1 - 2M/r \approx r'/2M + o(r'^2)$ Approximating $r^2 \approx (2M)^2$ and $r^4 \approx (2M)^4$.Using this approximation chatgpt gave us following solutions-

$$r(\tau) = 2M + \frac{\kappa^2}{8M}(\tau - \tau_0)^2 \tag{3.110}$$

$$t(\tau) = t_0 - \frac{16M^2 E}{\kappa^2(\tau - \tau_0)} \tag{3.111}$$

$$\theta(\tau) = \theta_0 + \frac{l_1}{4M^2}(\tau - \tau_0) \tag{3.112}$$

$$\phi(\tau) = \phi_0 + \frac{l_2}{4M^2 sin^2\theta} \tag{3.113}$$

[17] We are very thankful to chagpt for helping us entirely with it.

Here $\kappa^2 = E^2 - \frac{l_2^2}{4M^2 sin^2\theta}$. These set of solutions describe the near horizon motion in Schwarschild geometry.However we need to ensure that Pertubation theory for the solution is valid, this happens when $r'(\tau)$ is almost near to 2M. From equation 3.110 we find this happens when

$$\frac{\kappa^2}{8M}(\tau - \tau_0)^2 << 2M \tag{3.114}$$

Or

$$\tau - \tau_0 << \frac{4M}{\kappa} \tag{3.115}$$

So the timerange within which our solution remains valid is when equation 3.115 is satisfied.Here $\tau_0$ is assumed to be timescale when we start observing our system.This is alternative way of suggesting that there is distortion of timelike region near the horizon in a way that nearby points can either get repelled by angular momentum or may fall inside the horizon but can not remain stationary near the horizon for long time.Exact analysis requires solving the eliiptic integral expressions which we have not mentioned here for simplicity.If $\tau$ remains within this range then our solution is valid.If E is large $\kappa$ grows shortening the time interval before the perturbation grows too large.If $l_2$ is large we can see that it reduced $\kappa$ keeping the perturbation smaller for longer duration.This is like spinning the particle causes it to hover near the horizon escaping gravitational pull. So now we have the coordinates for Timelike region in terms of $\tau$.

### 3.10.1 Null diffeomorphism group for Schwarschild spacetime in near Horizon approximation for timelike region

Given set of equation from 3.110-3.113 and for timelike region as parametrized in section by equation 3.90 we have the following equation for Diffeomorphism symmetry matrix

$$\begin{bmatrix} \eta_0 \\ \eta_0 cos\theta_1 cos\phi_1 \\ \eta_0 cos\theta_1 sin\phi_1 \\ \eta_0 sin\theta_1 \end{bmatrix} = \begin{bmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{bmatrix} \begin{bmatrix} t_0 - \frac{16M^2E}{\kappa^2(\tau-\tau_0)} \\ 2M + \frac{\kappa^2}{8M}(\tau-\tau_0)^2 \\ \theta_0 + \frac{l_1}{4M^2}(\tau-\tau_0) \\ \phi_0 + \frac{l_2}{4M^2 sin^2\theta} \end{bmatrix} \tag{3.116}$$

Along withe condition that $Det(D) = 0$. With the parametrisation used in equation 3.32 -3.34 we can write the above the above equation as-

$$\begin{bmatrix} \eta_0 \\ \eta_0 cos\alpha_1 cos\beta_1 \\ \eta_0 cos\alpha_1 sin\beta_1 \\ \eta_0 sin\alpha_1 \end{bmatrix} = \begin{bmatrix} d_{11} & id_{22}cos\theta_2 cos\phi_2 & id_{33}cos\theta_3 cos\phi_3 & id_{44}cos\theta_4 cos\phi_4 \\ id_{11}cos\theta_1 cos\phi_1 & d_{22} & id_{33}cos\theta_3 sin\phi_3 & id_{44}cos\theta_4 sin\phi_4 \\ id_{11}cos\theta_1 sin\phi_1 & id_{22}cos\theta_2 cos\phi_2 & d_{33} & id_{44}sin\theta_4 \\ id_{11}sin\theta_1 & id_{22}sin\theta_2 & id_{33}sin\theta_3 & d_{44} \end{bmatrix} \tag{3.117}$$

$$\begin{bmatrix} t_0 - \frac{16M^2E}{\kappa^2(\tau-\tau_0)} \\ 2M + \frac{\kappa^2}{8M}(\tau-\tau_0)^2 \\ \theta_0 + \frac{l_1}{4M^2}(\tau-\tau_0) \\ \phi_0 + \frac{l_2}{4M^2 sin^2\theta} \end{bmatrix}$$

Since the equation for $\eta^T\eta = 0$ is universal and provided the 4 set of components of timelike vector are non zero and labelled sepeartely as distinct 4 vectors.We find these coefficients vanish again so for suitable parametrisation the algebra is still true and same as that of Minkowski spacetime.This algebra is **Grassman algebra and it is called Gr(0,4).**

### 3.10.2 Implications of Universality of Grassman Algebra

Since Gr(0,4) is a generic algebra of Null Diffeomorphism symmetry group elements it hints that null hypersurface contains fermionic degrees of freedom.Combined with coordinates or diffeomorphisms this implies a deep superspace structure to General relativity with or without supersymmetry.These hidden degrees of freedom may modify Black Hole Entropy counting and Hawking Radiation.This is in alignment with String Theory which suggests that spacetime has a graded superspace structure.Also the 4 grassman elements correspond to minimal supersymmetry.In D dimensions of spacetime the Null Difffeomorphism elements carry the Grassman algebra Gr(0,D) which seems to be a strong relation.As twistor theory formulates General relativity in terms of Spinors so Twsitor theory may be a good way to reformulate General relativity using Spinor Formalism as suggested by the algebra Gr(0,4).Since the algebra is universal with the diemnsionality of spacetime it has strong implications for known models of spacetime in Quantum gravity.As a last step our work so far needs to shed light on Null Hypersurface of Black Holes which seems to have fermionic degrees of freedom.This is our next step to proceed we substitute equation 3.117 into 3.97 to find Ricci scalar $R_\Sigma$ along Null Hypersurface.First thing to note that Grassman Algebra has real elements **so the Null Hypersurface is a complex manifold.** A matrix representation for 4 sets of generators is-

$$d_{11} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.118}$$

$$d_{22} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.119}$$

$$d_{33} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.120}$$

$$d_{44} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.121}$$

Equation 3.117 is cumbersome to proceed with the analysis we systematically rewrite equation 3.117 using new coordinates labelling each element in the timelike region as

$X_0, X_1, X_2, X_3$.Thus

$$X_0 = t_0 - \frac{16M^2E}{\kappa^2(\tau - \tau_0)} \tag{3.122}$$

$$X_1 = 2M + \frac{\kappa^2}{8M}(\tau - \tau_0)^2 \tag{3.123}$$

$$X_2 = \theta_0 + \frac{l_1}{4M^2}(\tau - \tau_0) \tag{3.124}$$

$$X_3 = \phi_0 + \frac{l_2}{4M^2 sin^2\theta} \tag{3.125}$$

Equation 3.117 then becomes-

$$\begin{bmatrix} \eta_0 \\ \eta_0 cos\alpha_1 cos\beta_1 \\ \eta_0 cos\alpha_1 sin\beta_1 \\ \eta_0 sin\alpha_1 \end{bmatrix} = \begin{bmatrix} d_{11} & id_{22}cos\theta_2 cos\phi_2 & id_{33}cos\theta_3 cos\phi_3 & id_{44}cos\theta_4 cos\phi_4 \\ id_{11}cos\theta_1 cos\phi_1 & d_{22} & id_{33}cos\theta_3 sin\phi_3 & id_{44}cos\theta_4 sin\phi_4 \\ id_{11}cos\theta_1 sin\phi_1 & id_{22}cos\theta_2 cos\phi_2 & d_{33} & id_{44}sin\theta_4 \\ id_{11}sin\theta_1 & id_{22}sin\theta_2 & id_{33}sin\theta_3 & d_{44} \end{bmatrix} \tag{3.126}$$
$$\times \begin{bmatrix} X_0 \\ X_1 \\ X_2 \\ X_3 \end{bmatrix}$$

Given equation 3.126 we notice that Null Hypersurface is also a complex Superspace element and that antiperodicity of grassmann elements and there squaring to zero implies that $\eta_0^2 = 0$.This is because from equation 3.126 we see that $\eta_0^2$ collects terms $d_{ii}^2$ and terms like $d_{ii}d_{jj} + d_{jj}d_{ii}$.These follow from the way the elements are squared and both of them are zero, the second one following from antisymmetry fo grassman elements.Hence the conclusion follows. Using equation 3.126 we can substitute for coordinates along null hypersurface and because of the grassmann nature of coordinates of null hypersurface we find that $R_\Sigma = 0$.Also notice that though $\eta_0^2 = 0$ hence expression like $cos^2\theta$ also diverge to infinity because $cos(\theta)$ goes inversely with $\eta_0$ hence $\cos^2\theta$ diverges because $\eta^0 = 0$. But what we want in equation 3.97 is the combined expression $(\eta_0 cos(\theta))^2$ .We can make similar arguments for terms in fourth powers in equation 3.97.Substituting for the expression from equation 3.126 we find that it simplifies to zero.Thus though some of the coordinates square to zero and some to infinity suggesting pathological coordinates but the combined terms can be substituted by taking limits appropriately and we find that the Ricci scalar for null hypersurface $R_\Sigma$ as given in equation 3.97 equals zero.To justify the arguments we write down the expression for $\eta_0$, $cos(\theta)$, $sin\theta$.Please note that expression for $sin\phi$, $\cos\phi$ always come with $\cos\theta$ or $\sin\theta$ hence if we take limits carefully our results does not change.Another way of saying so is that Ricci scalar $R_\sigma$ is entirely a function of null coordinates which are grassman variables and the expression for Ricci Scalar as in equation 3.97 involves terms which are square or higher powers of these variables on null hypersurface, hence it reduces to zero.However we will write the equations,the equations are-

$$\eta_0 = d_{11}X_0 + id_{22}cos\theta_2 cos\phi_2 X_1 + id_{33}cos\theta_3 cos\phi_3 X_2 + id_{44}cos\theta_4 cos\phi_4 X_3 \tag{3.127}$$

$$\tan\beta_1 = \frac{id_{11}cos\theta_1 cos\phi_1 X_0 + d_{22}X_1 + id_{33}cos\theta_3 sin\phi_3 X_2 + id_{44}cos\theta_4 sin\phi_4 X_3}{id_{11}cos\theta_1 cos\phi_1 X_0 + d_{22}X_1 + id_{33}cos\theta_3 sin\phi_3 X_2 + id_{44}cos\theta_4 sin\phi_4 X_3} \tag{3.128}$$

To find $\eta_0 cos\alpha_1$ we notice that we need to square the second and third equation coming from equation 3.126 and then take the square root of grassmann numbers.We do not know any procedure to take square root of grassmann numbers however we know that $(\eta_0 cos\alpha_1)^2 = 0$. But we can notice that by taking appropriate limits that $cos^2\alpha_1$ and $sin^2\alpha$ are finite.This is because $cos^2\alpha$ and $sin^2\alpha$ as perceived using equation 3.126 behave very similar to $tan^2\beta_1$.From equation 3.128 we notice that the square of both numerator and denominator is zero.Hence the suitable limits if taken provide a finite answer for $tan^2\beta_1$.Thus the grassman nature is not noticed for angular variables when the square is taken so our previous supposition of divergence of these angles when squared is not true.Since all the angular variables have square and fourth power dependence and are hopefully or necessarliy finite hence our proposition that $R_\Sigma = 0$ is true provided we take limits cautiously.**One of the implications of $R_\Sigma = 0$ is that the surface behaves like a minimal surface in higher dimensional geometry that is the Horizon satisfies Mean curvature constraints.** We notice that even if $R_\Sigma \neq 0$ then the dependence of Grassmannian nature is not observed for it. We can convince ourselves from the above equation that $\eta_0^2 = 0$ the reasoning is that terms that occur are of the form $d_{ii}^2$ and $d_{ii}d_{jj} + d_{jj}d_{ii}$ both of which are zero because of their Grassmann Algebra.Similarly by substitution from equation 3.126 we get expression for $\eta_0 cos\alpha cos\beta$ and we can convince ourselves again that it squares to zero. Similarly $\eta_0 cos\alpha_1 sin\beta_1$ also squares to zero and also using equation 3.126 we can see that $\eta_0 sin\alpha_1$ also squares to zero.We leave it as an exercise to the readers to verify that terms for Ricci scalar as provided in equation 3.97 only involve square terms of these grassmann variables. Equation 3.98 reduces to

$$-\frac{1}{deth}\frac{d^2 deth}{d\lambda^2} + \frac{1}{2(deth)^2}(\frac{d(deth)}{d\lambda})^2 = 0 \tag{3.129}$$

The solution to the above equation can be found as-

$$deth = (\gamma(\lambda - \lambda_0) + B)^2 \tag{3.130}$$

This is consistent with equation 3.99 being zero which implies n=0 or n=2. Using these ansatz we can solve equation 3.79 and find $\alpha$.We find $\alpha = 0$ or $\alpha = -\frac{5}{2}$.Interestingly we did not require gluing conditions to find $\alpha$.Using these ansatz we find that the expansion along Null hypersurface as per equation 3.64 has the form-

$$\theta' = \frac{\gamma}{\gamma(\lambda - \lambda_0) + B} \tag{3.131}$$

Using equation 3.62 we can compute the area of Null Hypersurface.However this involves a tedious computation of Grassman variables $\eta_0, \theta, \phi$.However if we use the interpretation of Expansion as rate of change of volume we find that the Area of Black hole Horizon has a parametric form-

$$A(\lambda) = Cexp(\lambda - \lambda_0 + D) \tag{3.132}$$

Equation 3.132 implies area of black hole increases with affine parameter $\lambda$ exponentially this is consistent with Hawking's Area Theorem and second law of Thermodynamics.We also notice that an explicit computation of Area A involves Grassmann Variables which has not been proceeded here.An explicit computation of entropy and its scaling with Area still needs to be verified however.

# 4 More imprecise hints to the Paradox

There are few more thought experiments suggesting the same conclusion-Suppose we consider an observer in Black Hole spacetime from a region A which is timelike and the observer falls inside the Black Hole along a null hypersurface as region B. Suppose we measure the line element of the observer $ds^2$ then $ds^2_{A\rightarrow B}$ represents the scenario where the observer falls inwwardly but the line element $ds^2_{B\rightarrow A}$ represents a scenario where the observer goes from a null region to timelike region which is not allowed by Causality.The reason it is not possible is because the Diffeomorphism matrix D is non invertible(as per equation 3.2) so there do not exists map from B to A.Thus the line element in this scenario is asymmetric which is not a feature of Riemannian geometry.Thus assuming there are disjoint regions like these in Black Hole Spacetime it does not necessarily admit a smooth differentiable map between such disjoint regions.So there can be many possibilities in this regard.However the asymmetry of line element is an indication of Black Hole Spacetime admitting a Finsler Geometry.We can see that metric is asymmetric in Finsler Geometry,however there can be many more possibilities in this regard.

## 4.1 Topology of Black Hole spacetime as a smooth foliation of 4 d Mobious strip but without ends of the strip being joined thus preserving Local Causal structure of Black Holes and with a Finsler geometry provided it has smooth gluing at of Cauchy data at the horizon and evades surgery at late times due to Gravitational blueshift ?

Suppose we consider a Cauchy slice embedded in $R^4$ then at a given instant of time we still find violation of Diffeomorphism symmetry but the coordinates here are not singular anywhere because $\eta$ still exists as a reliable coordinate chart. Maybe such foliations of spacetime do not exist.One reason for the same is because in presence of gravity different regions in space foliated as Cauchy slice experience different amount of redshift so if we follow 2 neighbhouring fibres of some base manifold due to Gravitational Redshift they undergo different amount of bending so they do not necessarily proceed as one parameter family of proper time $\tau$ as smooth fibres as we propogate local fiber evoluation data.Due to Gravitational redshift varying with time the fibres twist differently along a Cauchy slice.We call this the foliation problem in General Relativity.Usually the fibres studied in Differential geometry are assumed to be have a smooth Lie group structure or Lie algebra structure but because of bending of fibres we may not necessarily suppose that the symmetry of Lie group is preserved during foliation, the redhsift factor as we proceed along the radial direction along the base manifold appears to be different along nearby fibres of some base manifold. This suggests that the Non trivial bending of fibres may retain diffeomorphism

Invariance because as we see fibres away from a gravitating region bend less and fibres closer to horizon bend more.Suppose we consider a foliation of Cauchy slice then a horizontal Cauchy slice with a suitable time parametrization starts bending downwards closer to horizon as the time evolves but inside the horizon the Cauchy slice has undergone a 180 degree change so it is like a Mobious strip.The problem pictorially reduces to retaining Diffeomorphism invariance in a mobious strip but we are aware of Black hole complementarity as a feature of gravity.We suspect geometrically that the mobius strip generates singularity at the null hypersurface in some foliated slice.By suitably applying surgery theory techniques we can get disconnected manifolds.This suggessts a pictorial way of discontinuity of Cauchy data along such foliations but we rather insist in preserving the data on a single manifold as we do not want semiclassical physics to break down.We insist gluing these sets of data together as a problem in Quantum gravity.We do not know if the structure of a smooth manifold needs to be more scrutinized further to get relevant understanding of the problem.Certainly we need to presume a Finsler manifold structure on such spacetime even if we retain smoothness, though Diffeomorphism symmetry is already absent.We have 3 ingredients-Causal Strcuture of spacetime, Topological structure of spacetime, Smooth structure of spacetime but without Diffeomorphism.We retain Causality, Topology and Smooth structure of spacetime with no diffeomorphism but with a Finsler Geometry. We presume maybe the metric is smooth if we do not rely on tensorial nature of metric transformation.This means that the disjoint regions are combinations of different manifolds and precisely the non intutive bending of fibres and disjoint manifold regions together preserve the smooth nature of General relativity.We will not comment further about it. Also extrapolating the information from Null Hypersurface to Black Hole Interior again appears to require new ideas. We are worried that the absence of Diffeomorphism seems to imply towards Non Homegenous Manifolds or towards Finsler Geometry as eluded earlier(due to asymmetry in Line element.)Maybe Einstein Equations are valid without Diffeomorphism symmetry and there are meaningful notions of affine connection , Riemann Tensor but as a disjoint manifold.Are there geodesics in Black Hole Interior or on Null Hypersurface.We will not prefer saying anything here.Some open ill framed questions are-
1)Does bending of Fibres and Non trivial topological effects in spacetime manifold allow for Diffeomorphisms that are acausal precisely cancelling the 2 effects of Foliation problem and Diffeomorphism symmetry? Even if it is possible we state that it does not imply acausality? We will have more to say about it sometime later.

## 5 Black Hole Complementarity and Non Local Splitting of Hilbert Space

The non trivial bending of Cauchy slices foliated suitably seems to hint towards the possibility of Non local data evolution of base suitably embedded base manifold which is indicative of something similar to Black Hole Complementarity. In its most ideal form if we can provide diffeomorphisms due to shifting of fibres in a way that allows for acausal diffeomorphisms then certainly we have preserved Rienmannian structure of manifold at the cost of locality.Black hole complementarity also suggests a passaway from the asymmetry of line element at the cost of causality.Motivated by the principle of Holography we suggest another non intutive idea.Suppose Black Hole complementarity is a generic feature of Gravity which has been beautifully suggested in many papers of Professor Suvrat Raju and his collaborators, we find another paradox which has been satted by us in a pseudoprecise way.Suppose we have a hilbert space splitting of dual CFT [18] then the isomorphism of the Hilbert spaces in the bulk spacetime with the dual CFT implies allowing Black Hole Complemenatarity in bulk indicates something similar to complementarity in the hilbert spaces of dual CFT due to principle of Holography of information. This has not been stated in a very precise way.We are assuming an isomorphism of the splitted hilbert space in bulk with some kind of isomorphism in dual CFT but a feature like complementarity in dual CFT is an indication of some kind of non locality in CFT which seems to be non trivial.The inituition of associating isomorphism of this kind is known as subregion-subregion duality.Thus Black Hole complementarity, subregion-subregion duality consistency implies a non local ultraviolet completion of dual CFT.Puzzled by this paradox stated in an imprecise way we suggest that the firstgive away might be to assume smooth Riemannian structure of Black Hole spacetime with Finsler geometry. We will not comment anything more regarding this owing to our ignorance of the subject.

## 6 Conclusion

We reviewed Mathematical aspects of General Relativity in section 2.In section 3 we proposed a paradox owing to absence of diffeomorphism symmetry of Black Hole Spacetime in some small neighbhourhood ball region containing a timelike region and null region.We then used Raychaudhari equation and smooth variation of coordinate charts along such epsilon ball as its radious goes to zero.Hopefully it seems these non linear coupled ordinary differential equations and the smoothness of coordinate chart might give us a recipe for reconstructing the Null hypersurface and give us some light in the nature of Black Hole Spacetime.Interestingly such reconstruction for 2 D Minokowski spacetime seems to hint us towards matrix degrees of freedom of spacetime or alternatively the notion of superspace.Owing to complexity of Schwarschild spacetime we could not verify the argurment for Schwarschild spacetime and it will be interesting if Mathematicians , Physicist could shed more light in it by solving these set of equations suggested in section 3.6 and 3.7.We also saw that there are more imprecise hints to the geometry of Black Hole spacetime like Finsler Geometry as suggested in section 4 of the paper.In section 4.1 we also addressed the foliation problem due to Gravitational spacetime suggesting a notion of dynamically varying time field to suitably describe a Cauchy slice, we will refrain from making more comments regarding this now.Some problems that are of interest for further investigation are-

1)Connection of Entropy with geometry using a generic framework.

2)Is Area Law of Black Holes verifiable using a different approach as proposed.

Investigating Quantum effects , the paper persues ideas using purely Classical Franework.

3) Problem of Foliation as a probe for dimensional reduction which needs justification.Investigating back reaction effects which may be significant at later times.

[18]Assuming some non perturbative completion of the ultraviolet regime of CFT preserves the divergences

4) Connections with Celestial Holography, Asymptotic symmetries and String Theory.
5) Finding an explicit entropy counting of Black Hole in this framework if possible.

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