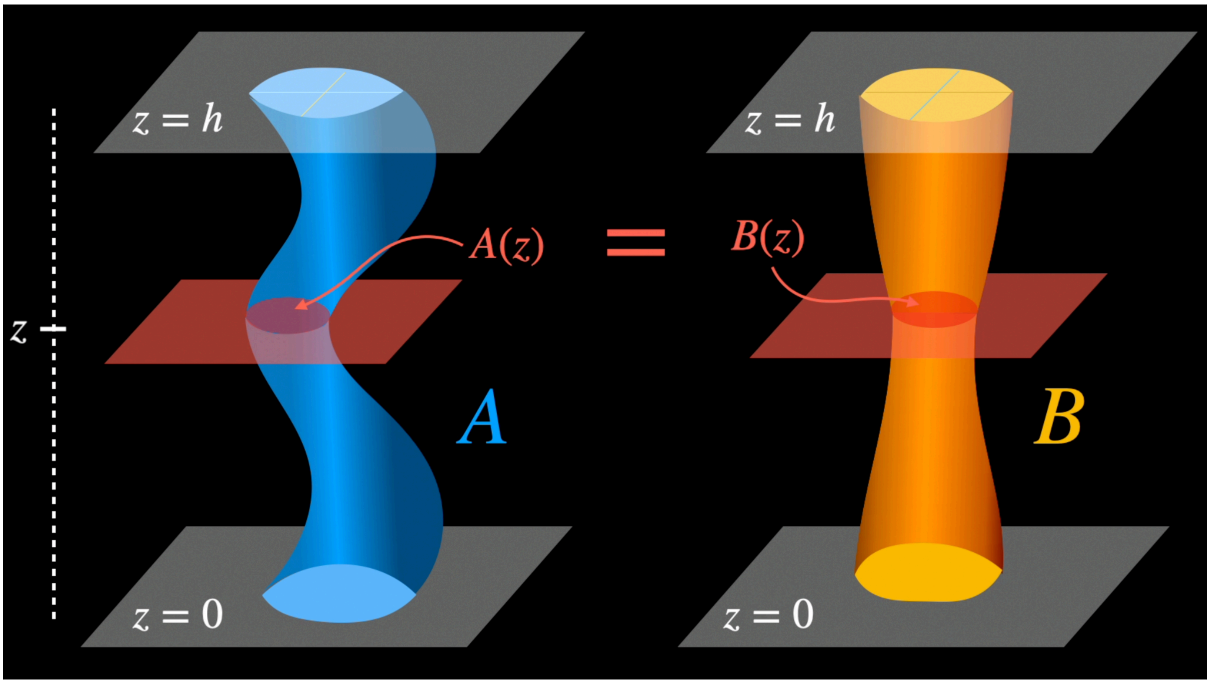
Cavalieri's Principle

Definition: *If two solids A and B are such that for every height z from a reference plane, the cross-sectional area of A at height z is equal to the cross-sectional area of B at the same height z, then the volumes of A and B are equal.*

$$\mathrm{Vol}(A) = \int_0^h A(z)\,dz \quad \text{and} \quad \mathrm{Vol}(B) = \int_0^h B(z)\,dz$$

If $A(z) = B(z)$ for all $z \in [0, h]$, then

$$\mathrm{Vol}(A) = \mathrm{Vol}(B)$$



Proof

Since the cross-sectional areas of solids A and B are equal at every height z, we have $A(z) = B(z)$ for all z.

Thus, the integrals of these areas over the same height h will also be equal:

$$\mathrm{Vol}(A) = \int_0^h A(z)\,dz = \int_0^h B(z)\,dz = \mathrm{Vol}(B)$$

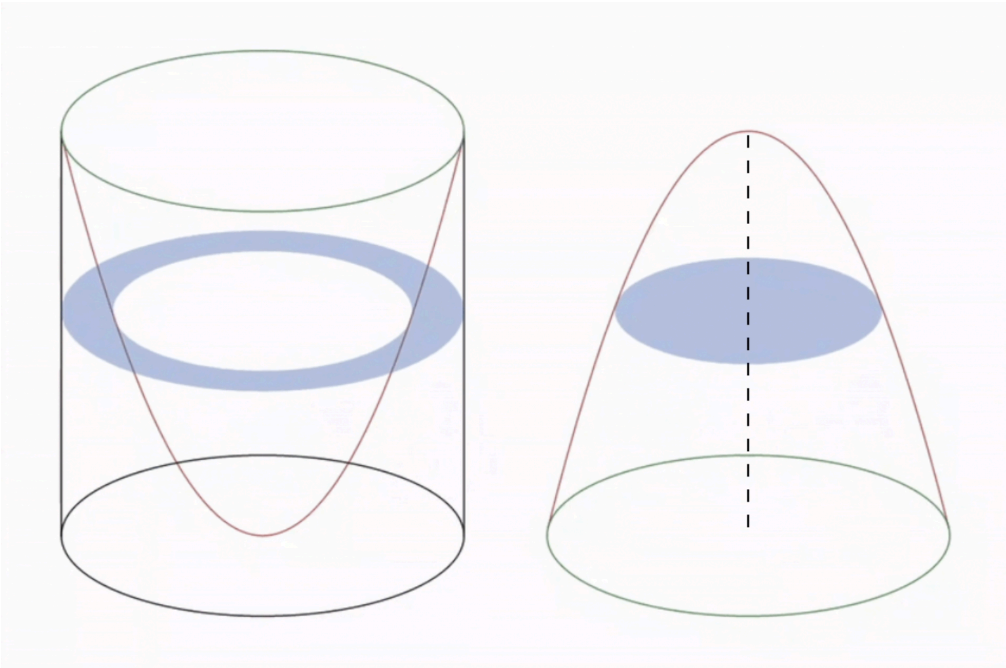
Therefore, by Cavalieri's Principle, the volumes of the two solids A and B are equal.

Q.E.D.

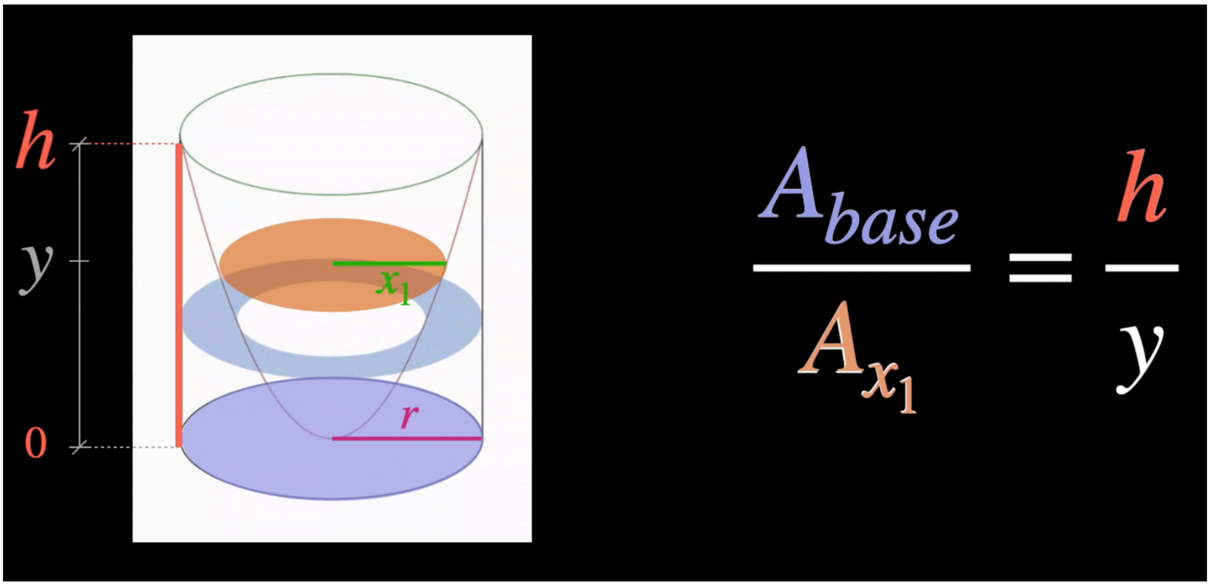
Paraboloids

We want to prove that the following two solids have the same cross-sectional areas for all heights y between *zero* and h. This will allow us to apply Cavalieri's Principle in order to conclude that their volumes are the same, without further calculations.

The left one is a cylinder with a paraboloid carved out of it. The right one is simply the paraboloid that was carved out from the solid on the left, and placed upside down.



Let's study the left one first. Notice that its cross-sections are rings.

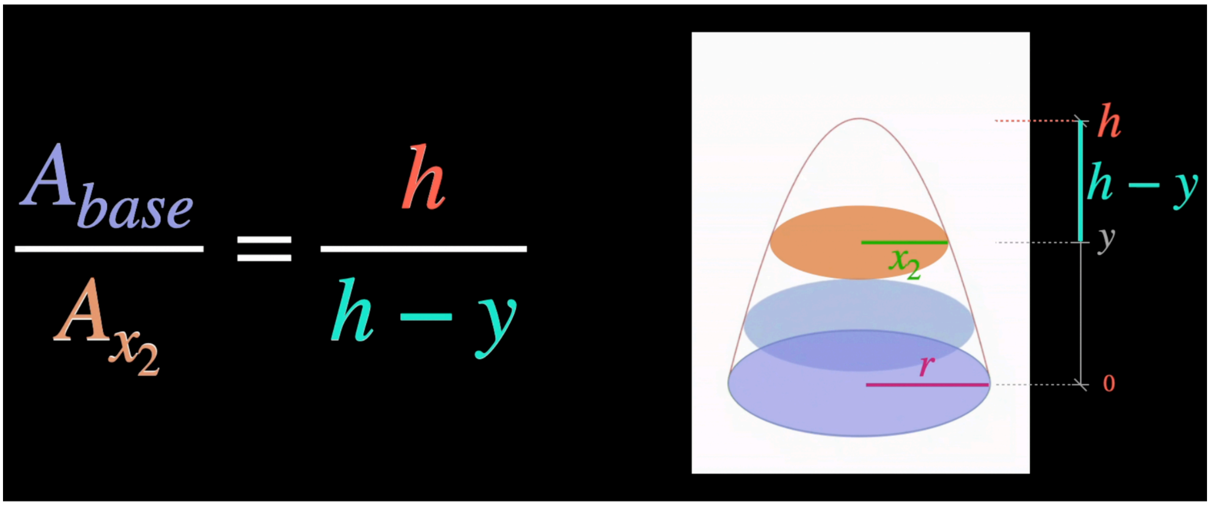


$$\frac{A_{\text{base}}}{A_{x_1}} = \frac{h}{y}$$

Here, A_{base} is the area of the base of the cone, and A_{x_1} is the area of a cross-section of the cone at height y from the base. r is the radius of the base, and x_1 is the radius of the cross-sectional circle at height y.

$$\frac{\pi r^2}{\pi x_1^2} = \frac{h}{y} \quad \Longrightarrow \quad y = h\left(\frac{x_1}{r}\right)^2 \quad \Longrightarrow \quad \boxed{x_1 = r\sqrt{\frac{y}{h}}}$$

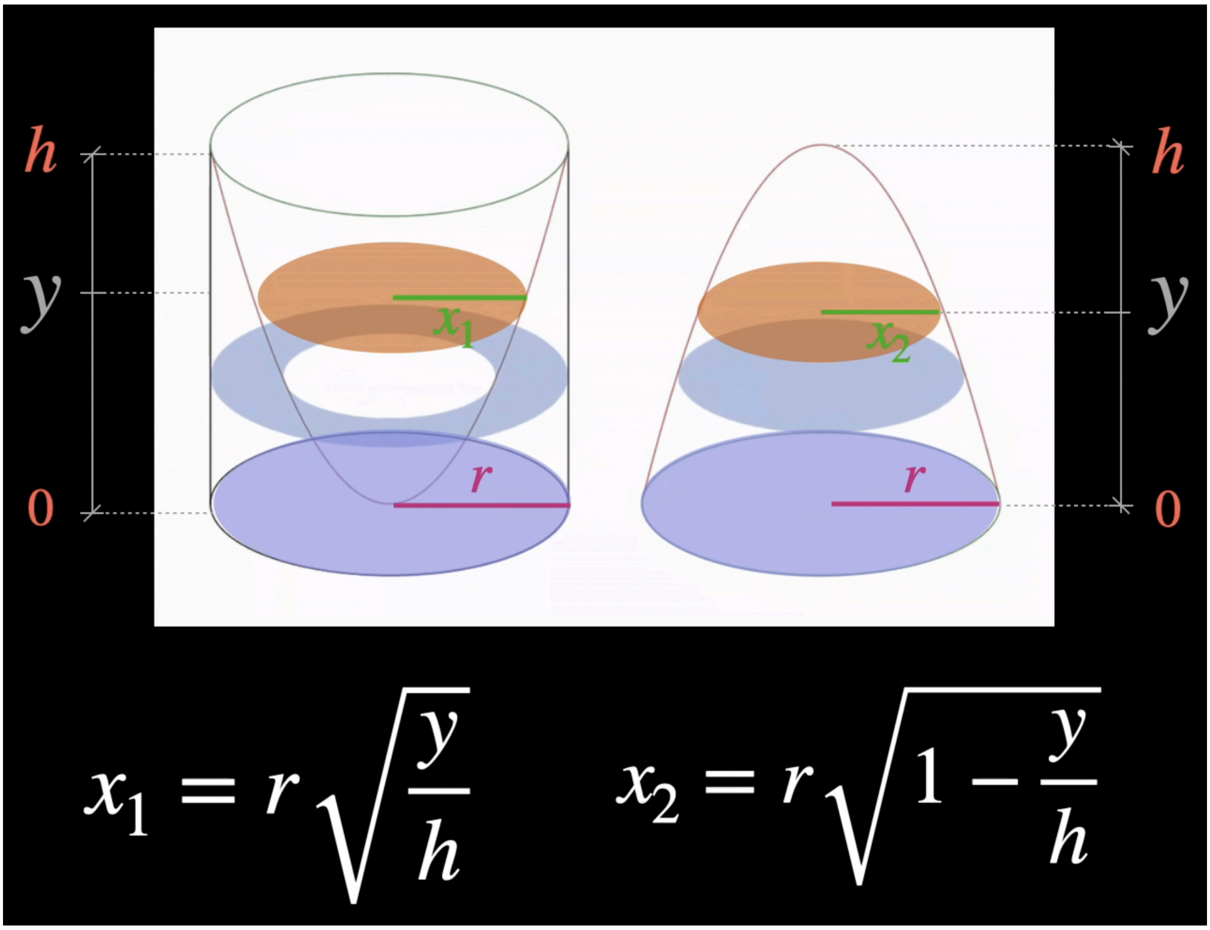
Now we will study the solid on the right. Notice that its cross-sections are disks.



$$\frac{A_{\text{base}}}{A_{x_2}} = \frac{h}{h-y}$$

Here, A_{base} is the area of the base of the cone, and A_{x_2} is the area of a cross-section at a height $h-y$ from the top of the cone. r is the radius of the base, and x_2 is the radius of the cross-sectional circle at this height.

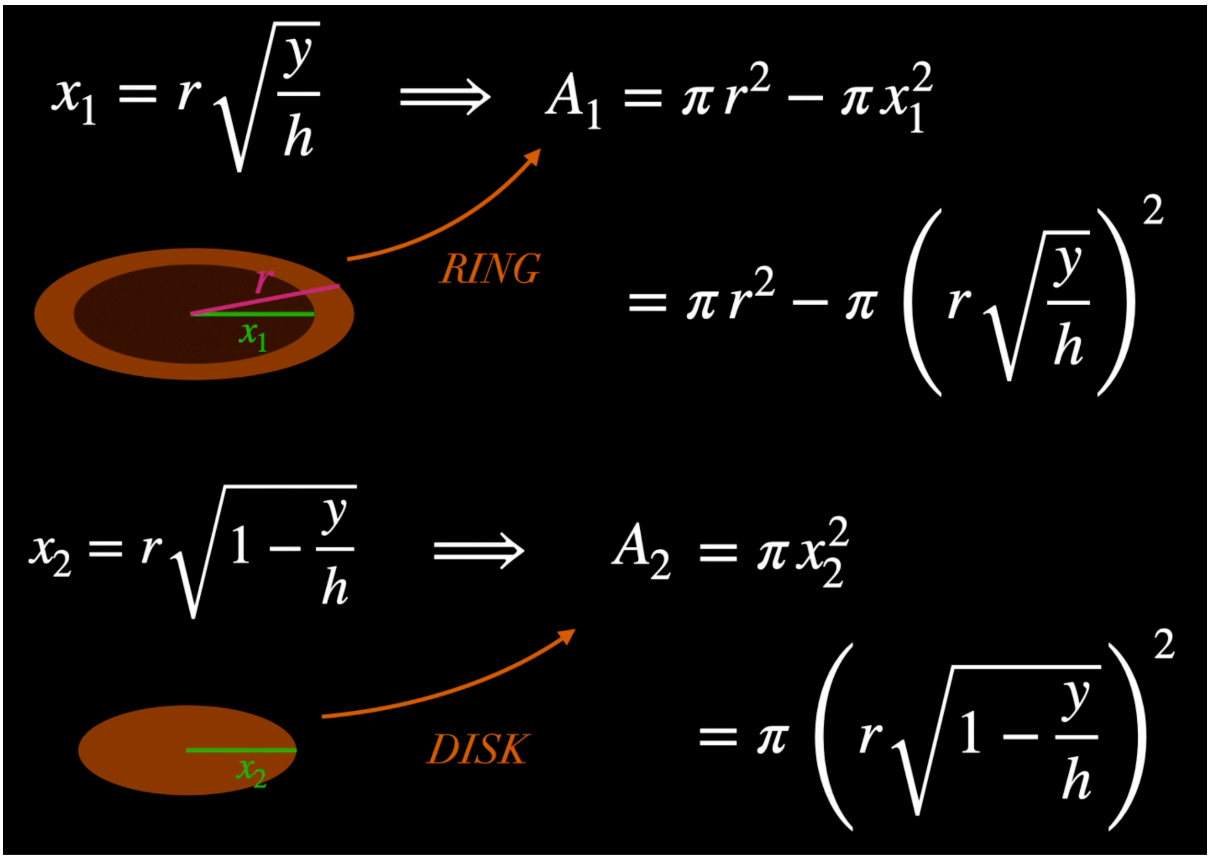
$$\frac{\pi r^2}{\pi x_2^2} = \frac{h}{h-y} \quad \Longrightarrow \quad y = h - h\left(\frac{x_2}{r}\right)^2 \quad \Longrightarrow \quad \boxed{x_2 = r\sqrt{1-\frac{y}{h}}}$$



$$x_1 = r\sqrt{\frac{y}{h}} \quad \Rightarrow \quad A_1 = \pi r^2 - \pi x_1^2 \;= \pi r^2 - \pi\left(r\sqrt{\frac{y}{h}}\right)^2$$

$$x_2 = r\sqrt{1-\frac{y}{h}} \quad \Rightarrow \quad A_2 = \pi x_2^2 \;= \pi\left(r\sqrt{1-\frac{y}{h}}\right)^2$$

We found relations between the horizontal lengths (x_1 and x_2) and the variable heights *y*. This allows us to calculate their cross-sectional areas.



Rewriting A_2 in a more convenient way, we find out the following:

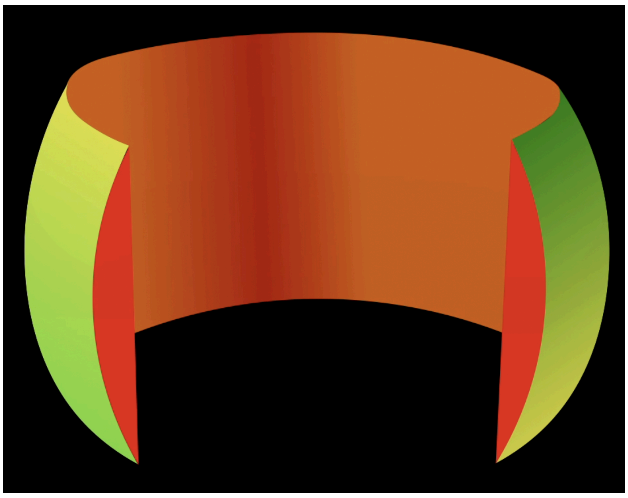
$$A_2 = \pi \left(r\sqrt{1-\frac{y}{h}}\right)^2 = \pi \left(1-\frac{y}{h}\right) r^2$$

$$= \pi r^2 - \pi \left(\sqrt{\frac{y}{h}}\right)^2 r^2 = A_1$$

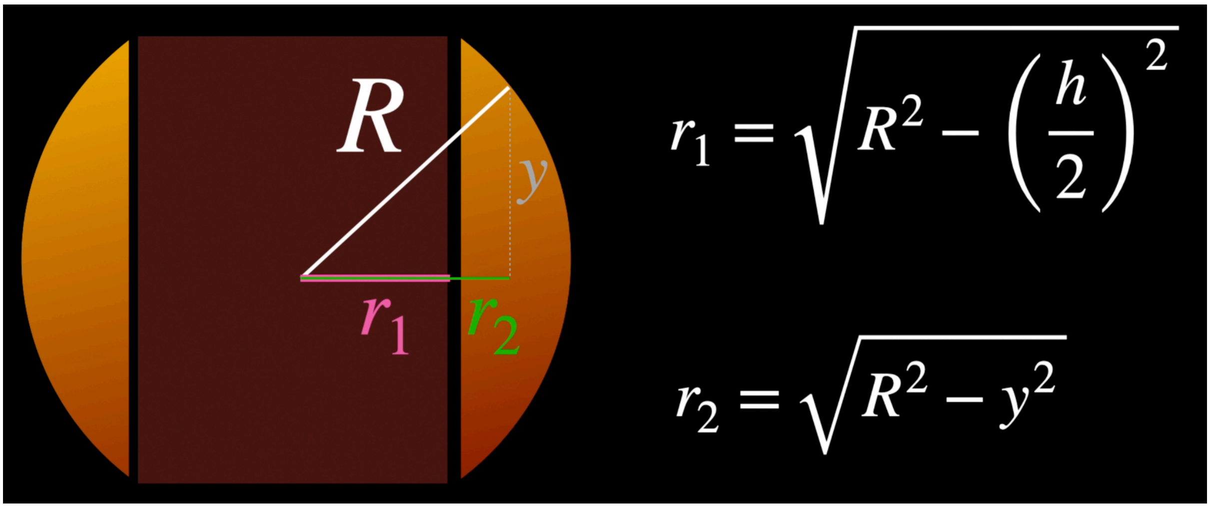
Hence, we found out that their cross-sectional areas, for any *y* between *zero* and *h*, are indeed equal. And therefore, Cavalieri's Principle allows us to conclude that their volumes are equal as well.

The Napkin Ring Problem

Assume a right circular cylinder whose axis goes through the center of a sphere with radius *R*. Let *h* be the height of the cylinder segment that is within the sphere. This band is the region of the sphere that lies outside the cylinder.



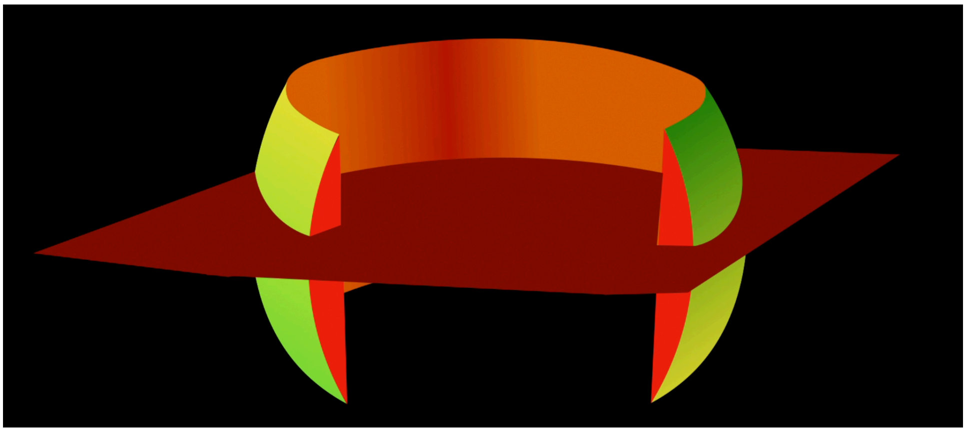
There are 3 radii involved here:



$$r_1 = \sqrt{R^2 - \left(\frac{h}{2}\right)^2}$$

$$r_2 = \sqrt{R^2 - y^2}$$

Now we can calculate the cross-sectional areas.



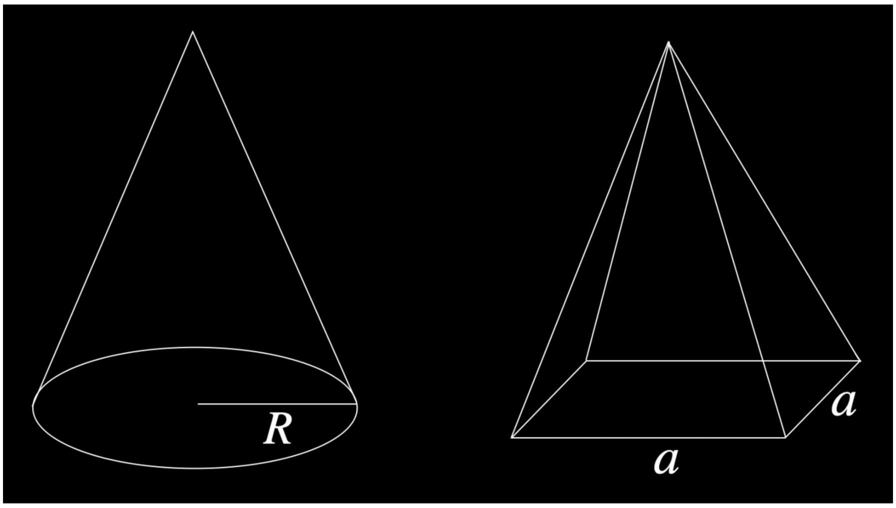
$$A=\pi r_2^2-\pi r_1^2=\pi\left(\sqrt{R-y^2}\right)^2-\pi\left(\sqrt{R-\left(\frac{h}{2}\right)^2}\right)^2=\pi\left(\frac{h^2}{4}-y^2\right)$$

We notice that the cross-sectional areas do not depend on the radius *R* of the sphere! I.e. it doesn't matter if this band goes around the Earth, or around a basketball, their volumes are the same.

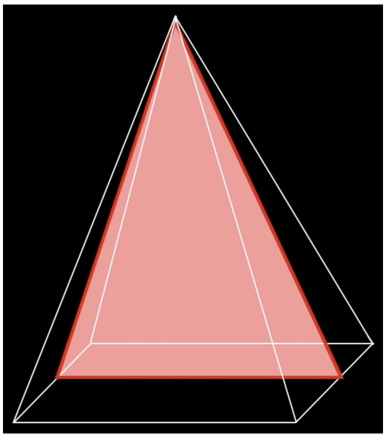
$$V_{\text{band}}=\int_{-\frac{h}{2}}^{\frac{h}{2}}A\,dy=\int_{-\frac{h}{2}}^{\frac{h}{2}}\pi\left(\frac{h^2}{4}-y^2\right)dy=\frac{\pi h^3}{6}$$

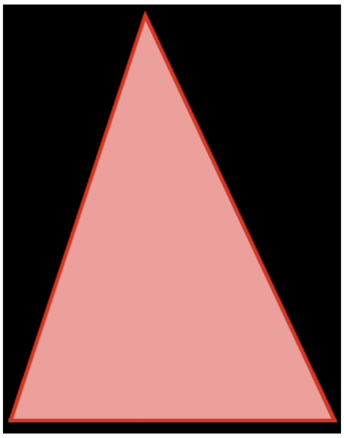
Pyramid & Cone

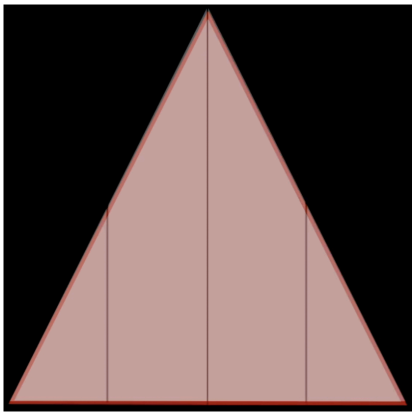
We will make the pyramid with square base and the cone to have the same cross-sectional areas for all heights *y* between *zero* and *h*. This will be accomplished by finding a condition between them.

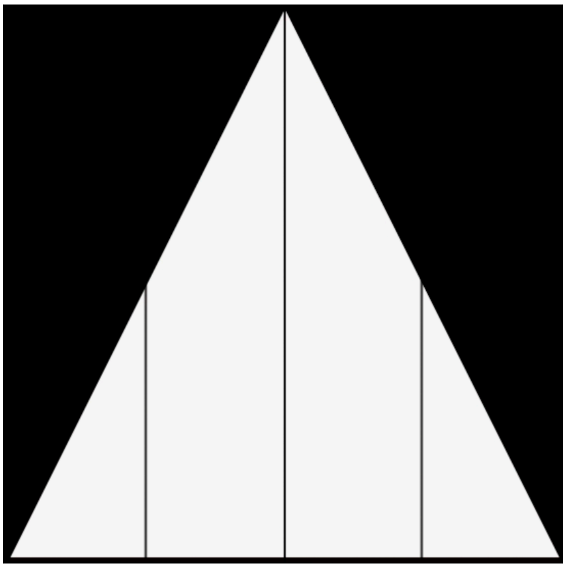


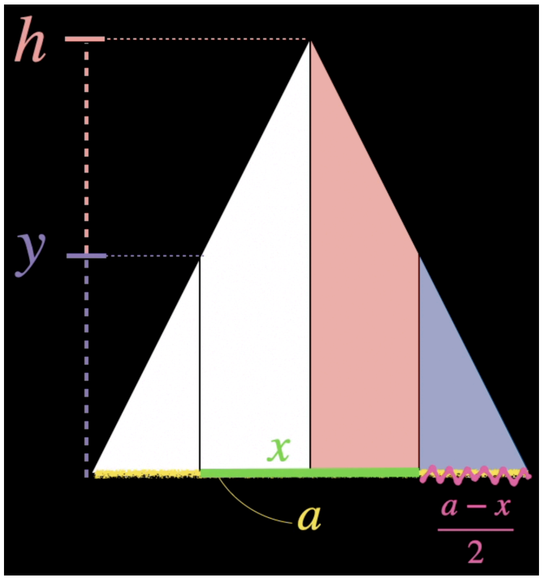
Let’s focus on the pyramid first. These are the lengths that will be defined on this triangle:











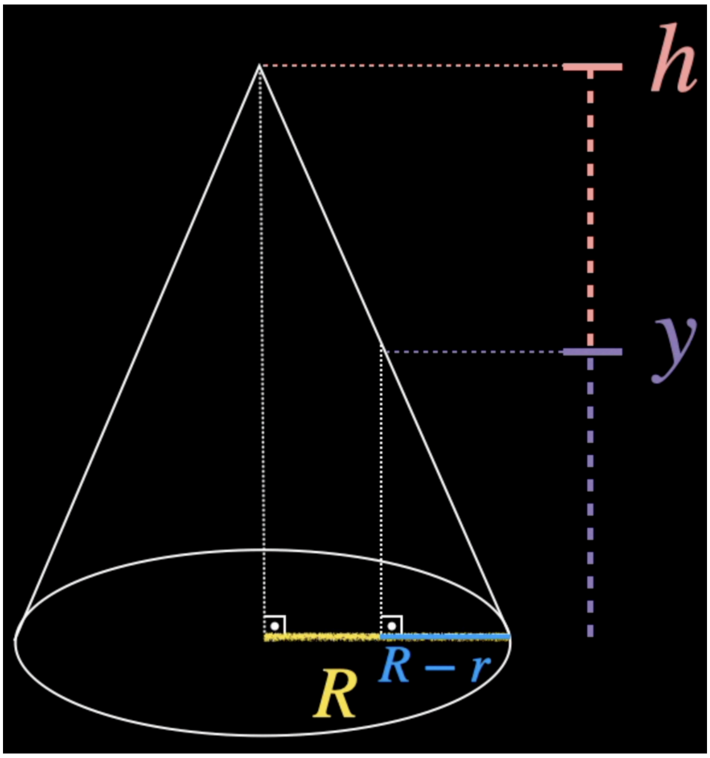
$$\frac{\text{height}}{\text{hypotenuse}} = \frac{h}{\sqrt{h^2 + \left(\frac{a}{2}\right)^2}} = \frac{y}{\sqrt{y^2 + \left(\frac{a-x}{2}\right)^2}}$$

This relation implies the following:

$$\frac{h}{\sqrt{h^2 + \left(\frac{a}{2}\right)^2}} = \frac{y}{\sqrt{y^2 + \left(\frac{a-x}{2}\right)^2}} \quad \Rightarrow \quad y^2\left(h^2 + \frac{a^2}{4}\right) = h^2\left(y^2 + \left(\frac{a-x}{2}\right)^2\right)$$

$$\Rightarrow \quad (\ldots) \quad \Rightarrow \quad \boxed{x = a\left(1 - \frac{y}{h}\right)}$$

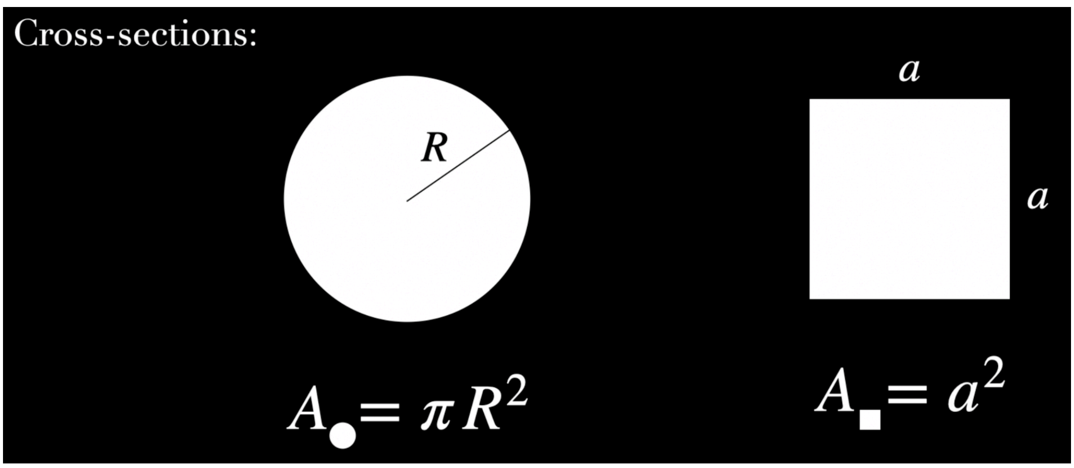
Thus, we found *x* as a function of the height *y*.
Now we turn to the cone.



$$\frac{\text{height}}{\text{hypotenuse}} = \frac{h}{\sqrt{h^2 + R^2}} = \frac{y}{\sqrt{y^2 + (R - r)^2}}$$

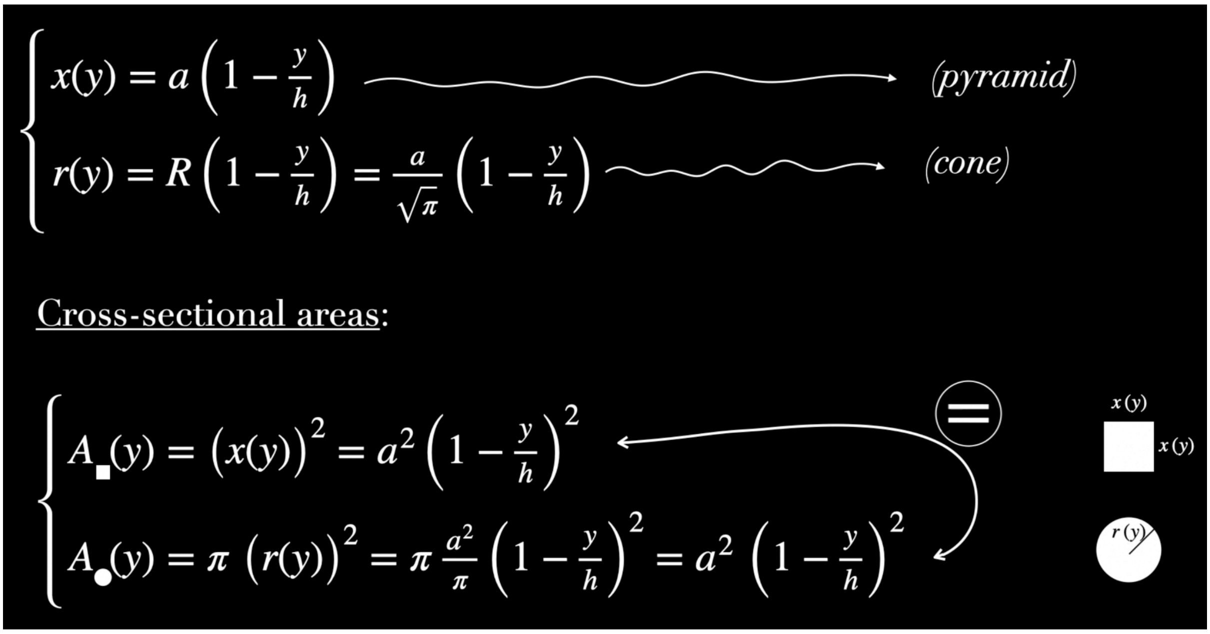
$$\Rightarrow \quad (\dots) \quad \Rightarrow \quad \boxed{r = R\left(1 - \frac{y}{h}\right)}$$

Now we find the condition related to their bases.



$$A_{\bigcirc} = \pi R^2 \quad \text{and} \quad A_{square} = a^2$$

$$\Rightarrow \quad \pi R^2 = a^2 \quad \Rightarrow \quad \boxed{R = \frac{a}{\sqrt{\pi}}}$$



The next step is to put everything together and notice that their cross-sectional areas are indeed equal.

$$\begin{cases} x(y) = a\left(1 - \frac{y}{h}\right) & \text{(pyramid)} \\ r(y) = R\left(1 - \frac{y}{h}\right) = \frac{a}{\sqrt{\pi}}\left(1 - \frac{y}{h}\right) & \text{(cone)} \end{cases}$$

Cross-sectional areas:

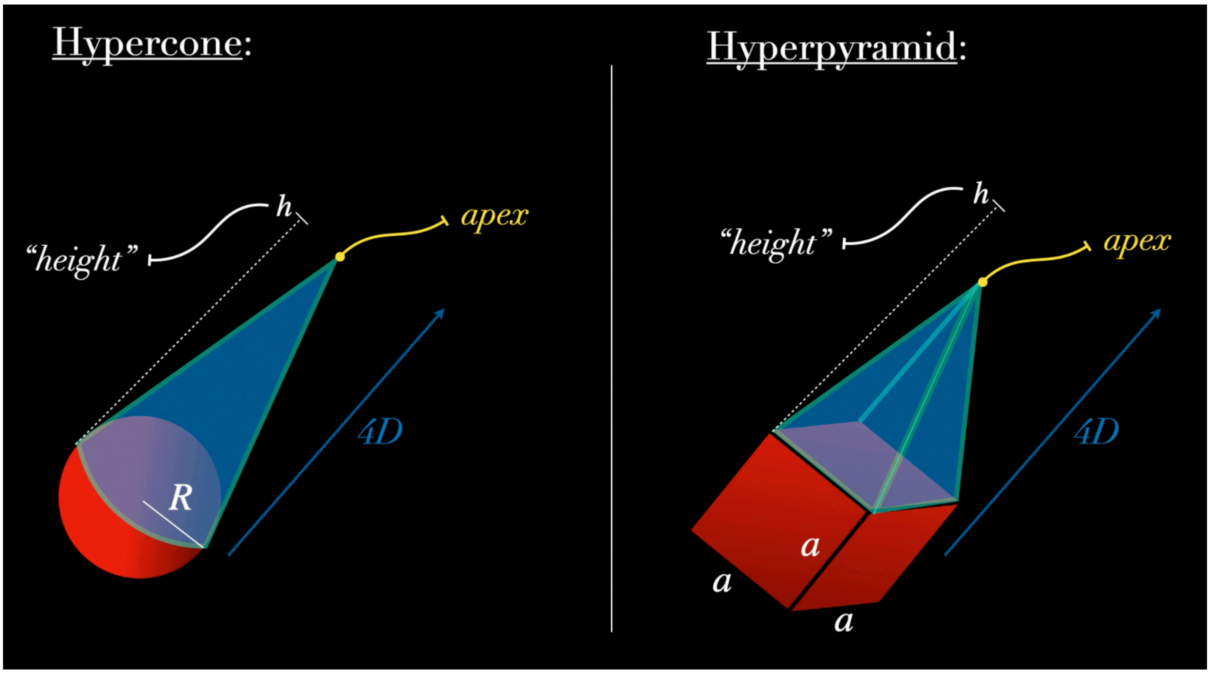
$$\begin{cases} A_{square}(y) = (x(y))^2 = a^2\left(1 - \frac{y}{h}\right)^2 \\ A_{\bigcirc}(y) = \pi(r(y))^2 = \pi\frac{a^2}{\pi}\left(1 - \frac{y}{h}\right)^2 = a^2\left(1 - \frac{y}{h}\right)^2 \end{cases}$$

We can finally use Cavalieri's Principle to conclude that their volumes are the same as well.

$$\Downarrow \quad \text{(Cavalieri's Principle)}$$

$$V_{\text{cone}} = \frac{1}{3}\pi R^2 h = \frac{1}{3}\pi\left(\frac{a}{\sqrt{\pi}}\right)^2 h = \frac{a^2 h}{3} = V_{\text{pyramid}}$$

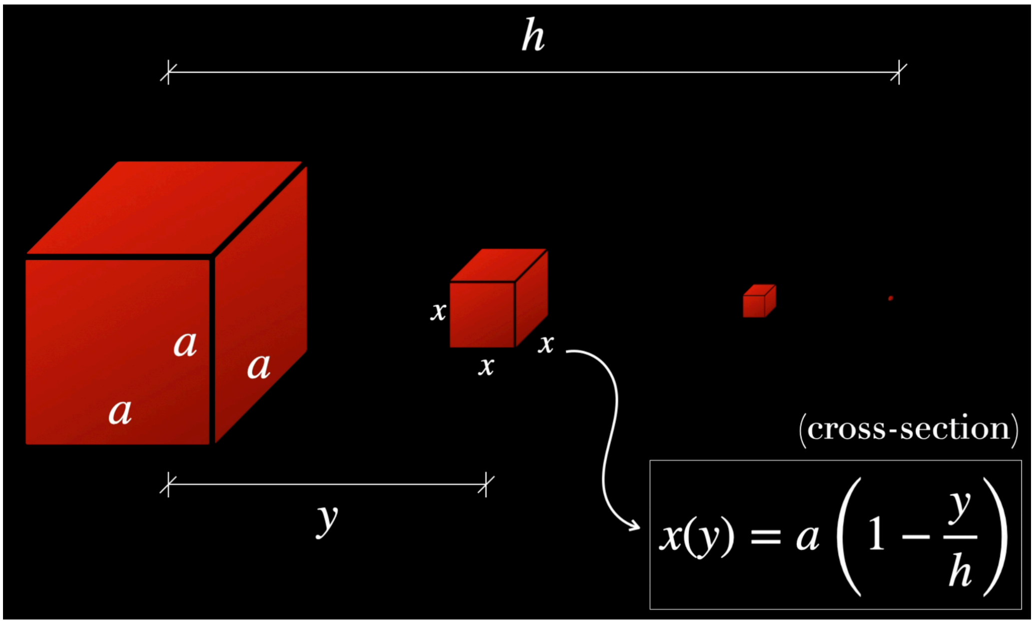
Pyramid & Cone in 4D



Cavalieri's Principle "in 4D":

"If two 4-dimensional solids have the same 3-dimensional 'cross-sectional' volume at every 'height' along the fourth dimension, then their 4-dimensional hypervolumes are equal."

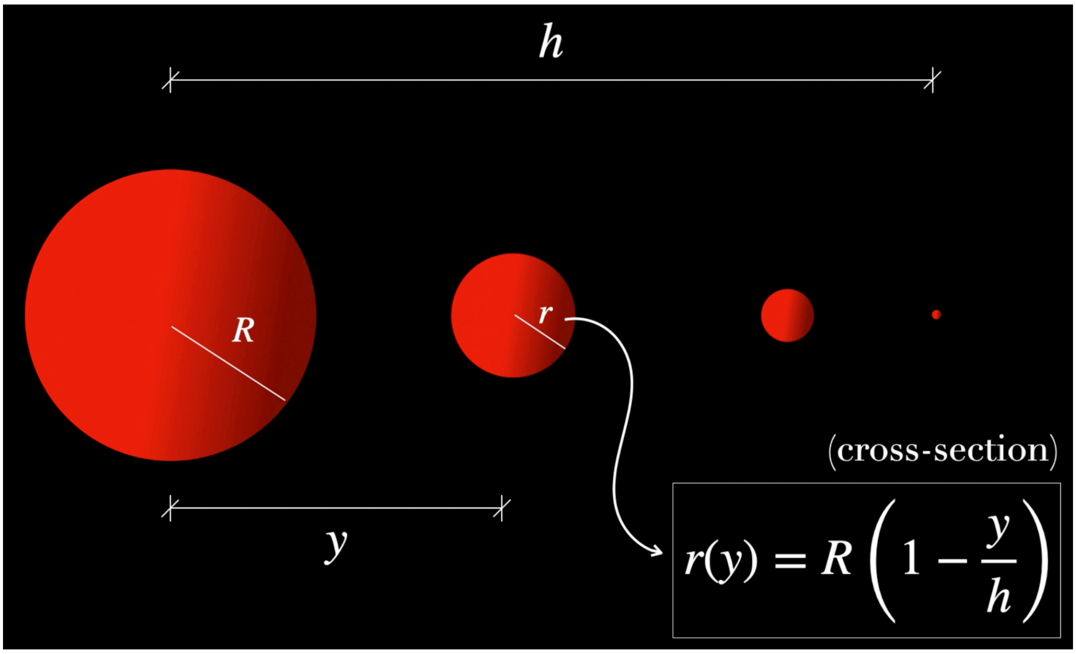
For the hypercube, we have the following:



$$x(y) = a\left(1 - \frac{y}{h}\right)$$

This expression is the side of a cubic “cross-section” of the hyperpyramid as a function of the “height” *y* along the fourth dimension.

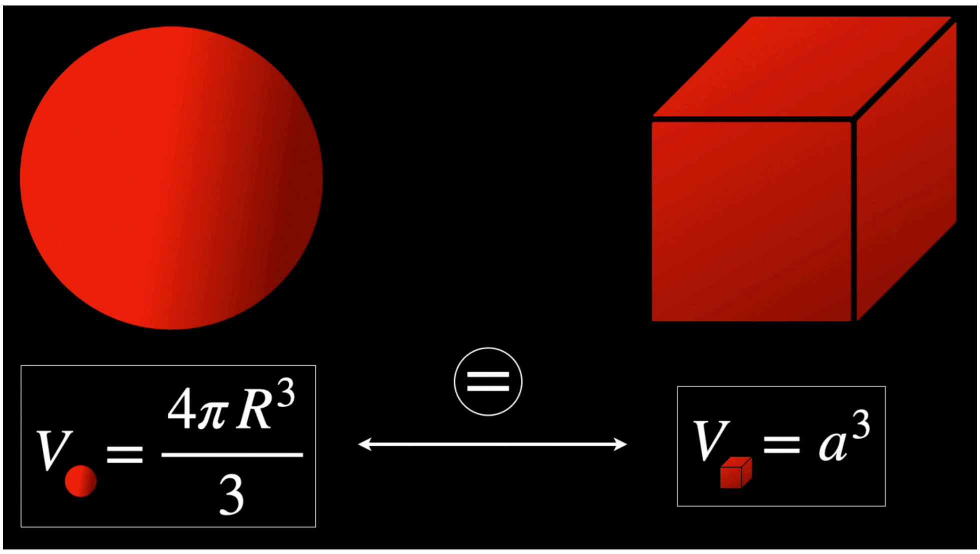
Meanwhile, for the hypercone, we have the following:



$$r(y) = R\left(1 - \frac{y}{h}\right)$$

Similarly, the expression above is the radius of a spherical “cross-section” of the hypercone as a function of the “height” *y* along the fourth dimension.

We need to make sure that the two volumes, which are the “cross-sections” now, are the same, so that we can apply Cavalieri’s Principle.



$$\boxed{V = \frac{4\pi R^3}{3}} \quad \Leftrightarrow \quad \boxed{V = a^3} \quad \Rightarrow$$

$$\Rightarrow \quad R^3 = \frac{3a^3}{4\pi} \quad \Rightarrow \quad \boxed{R = a\sqrt[3]{\frac{3}{4\pi}}}$$

Putting everything together now, we make sure that the condition above implies equal volumes, just as required.

$$\begin{cases} x(y) = a\left(1 - \frac{y}{h}\right) \\ r(y) = a\sqrt[3]{\frac{3}{4\pi}}\left(1 - \frac{y}{h}\right) \end{cases}$$

$$\begin{cases} V_{square}(y) = a^3\left(1 - \frac{y}{h}\right)^3 \\ V_{\bigcirc}(y) = \frac{4\pi}{3}a^3\left(\frac{3}{4\pi}\right)\left(1 - \frac{y}{h}\right)^3 = a^3\left(1 - \frac{y}{h}\right)^3 \end{cases}$$

And thus, Cavalieri’s Principle implies that their hypervolumes are the same as well.

$$\mathcal{H}_{\text{hyperpyramid}} = \frac{1}{4}V_{\text{base}}\,h$$

$$= \frac{1}{4}a^3\,h = \mathcal{H}_{\text{hypercone}} \qquad \text{Q.E.D.}$$

One could easily generalize it to higher dimensions as well. For *n*-dimensional solids (like a hyperpyramid and a hypercone in *n*D), the *(n-1)*-dimensional “cross-section” would have the following hypervolumes:

$$\boxed{V_{\text{hypercube}}(y) = a^n\left(1 - \frac{y}{h}\right)^n = V_{\text{hypersphere}}(y)}$$

For the *(n-1)*-hypersphere, that is the base of the *n*D hypercone, we would have that:

$$r(y) = a\,(?)^{\frac{1}{n}}\left(1 - \frac{y}{h}\right)$$

And the condition on the radius *R*, with respect to the side *a*. $R = R(a) = \ldots ?$

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