Calculus on Manifolds

Take a space, which looks like a bunch of rectangles stitched together. That's a Euclidean space, much like the intuitive kind we experience around us.

But if we zoom out, we see that these stitched rectangles are actually a sphere, which exists in a 3 dimensional space. The sphere is a manifold

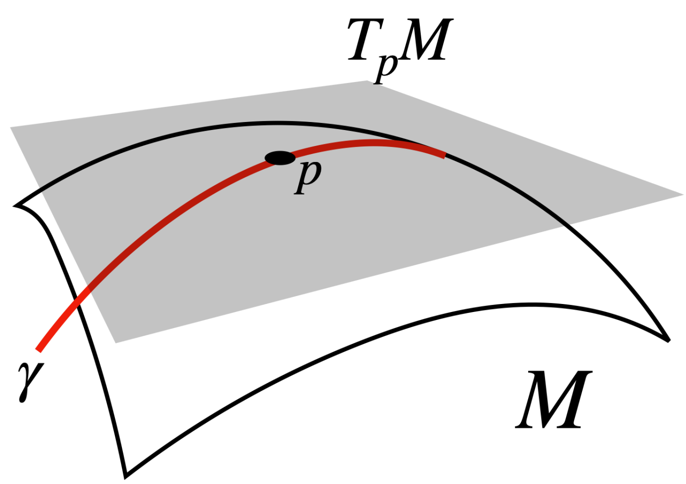
Because we can go back and forth between these states with ease, from a flat plane to a curved ball, they are locally homeomorphic.

This is a crucial property – because, there are certain spaces where, if you zoom in on them, they do not resemble a Euclidean space, and therefore cannot be manifolds.

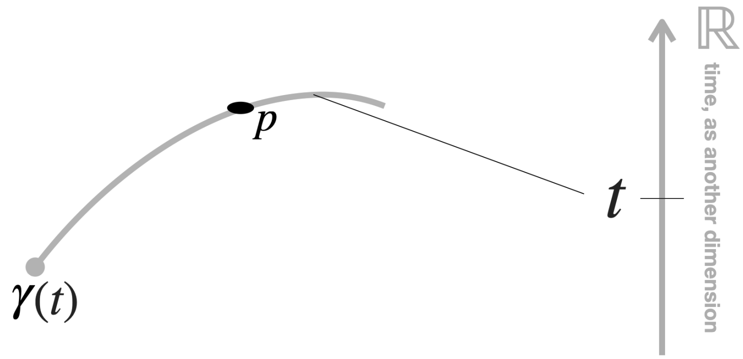
What can we do with a manifold? Say, we want to calculate things like distance on the curve of the manifold.

We have a manifold *M*. In order to measure the distance of the curve, we need to use a tangent space for a point *p* on the manifold.

Say we trace a specific path on *M*, called γ through the point *p*. You can think of γ as the trajectory of a moving point.

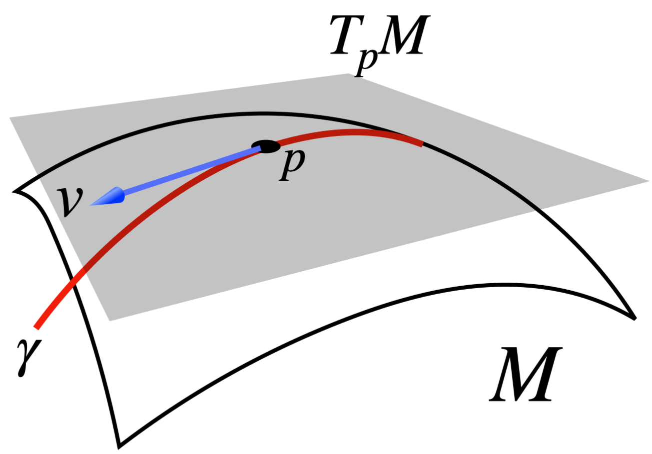


This path is parameterized by t, meaning t is a variable (like time) that allows us to describe the position of the point on the path at any given moment.



As this curve \gamma moves through p, its "velocity" at p isn't just about speed but also direction—it's where and how fast the curve is moving at that exact point.

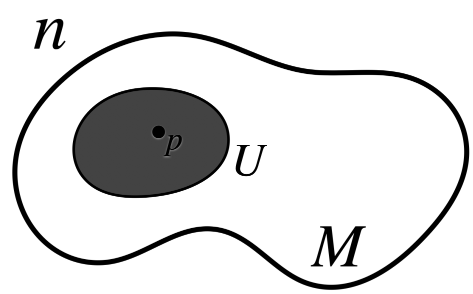
This creates a tangent vector, located precisely at point p, which for now we will call v.



This figure is a bit misleading, because the tangent space/vector doesn't necessarily look literally like a plane tangent to the manifold – it’s something abstract. It *can* look like this when it is embedded in a higher dimension space, like we have here, which helps us visualize it. Essentially this entire explanation helps us develop a good intuition of things.

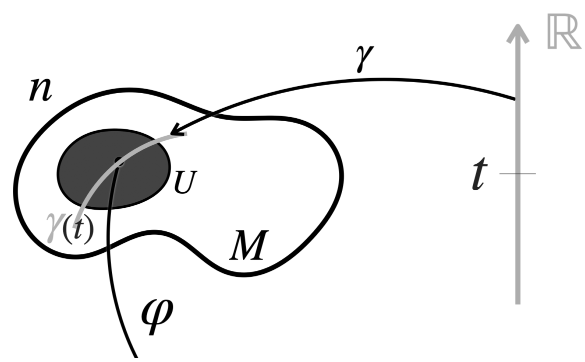
Manifolds can also exist in higher dimensions like n. How can we imagine them? We can “zoom in” on them, not in the literal sense, through applying functions on a particular point on that manifold, which we choose, and work on it in the intuitive realm of a Euclidean space.

We have a manifold *M*, but in order to pinpoint a point, we need to create a neighborhood of points called *U*, an open set, where the point *p* is floating freely.



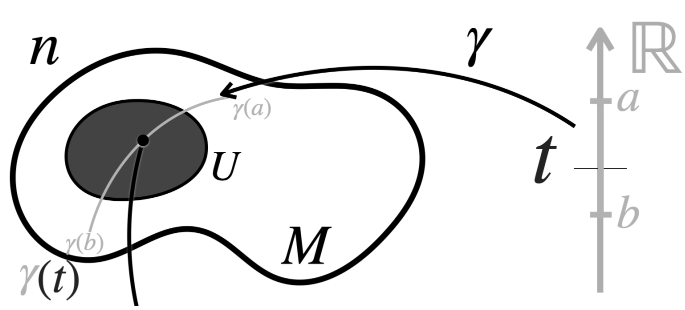
Mathematically, this is represented as $\varphi: U \rightarrow \mathbb{R}^{n}$ where φ is a coordinate map that maps *U* homeomorphically to an open subset of Euclidean space $\mathbb{R}^{n}$, meaning that it preserves the structure of the manifold locally.

On the original manifold, we have a curve that runs along it, γ, where *t* like time, or any other continuous variable, is the parameter.



It's represented like this $\gamma: t \rightarrow M$ and is read like "gamma is a map from t to M". As we established previously, for each value of *t*, $\gamma(t)$ gives a point on the manifold. In this curve, *t* ranges from the initial value *a* and the final value *b*.

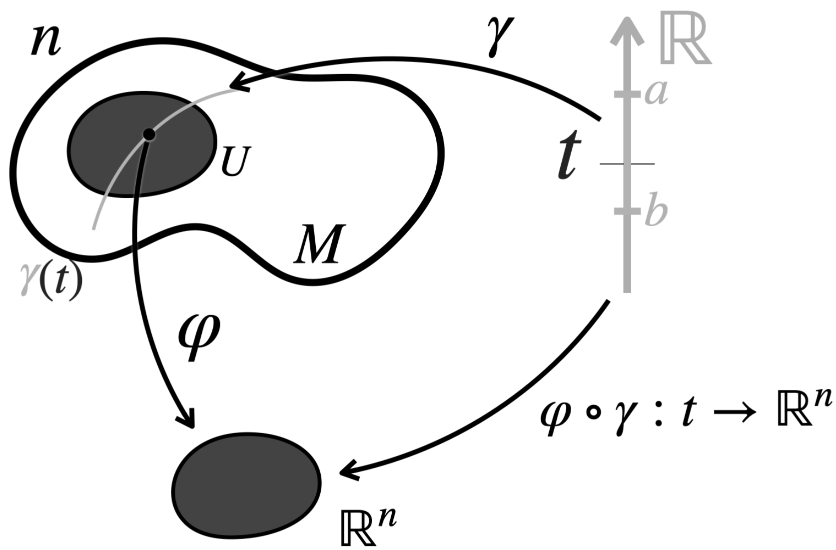
$$t \in [a, b] \text{ to } M$$



Now, we want to imagine that we are walking along this curve, but not on M directly, but on the *local* coordinates.

We take the $\gamma(t)$, the curve on the manifold *M*, and φ, the coordinate map that takes points from the manifold and maps them to local coordinates in $\mathbb{R}^n$.

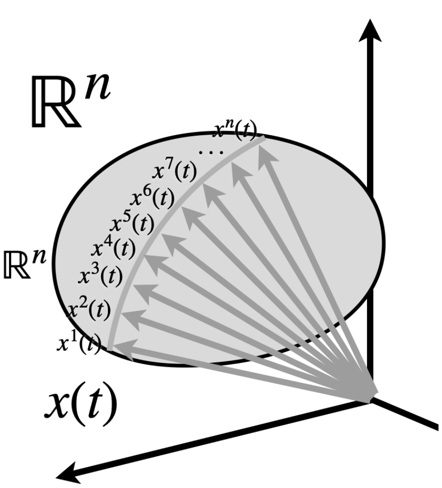
This gives us a *composite function*, made of combining 2 functions together.



The function $\varphi \circ \gamma$ takes the parameter *t*, and gives you a vector of coordinates in $\mathbb{R}^n$ (or the local Euclidean space).

After applying the map φ, the curve $\gamma(t)$ on the manifold is now represented as *x(t)* in local coordinates. The curve $\gamma(t)$ on the manifold *M* is now represented in terms of local coordinates.

$$x(t) = \left(x^1(t), x^2(t), \ldots, x^n(t)\right) \text{ in } \mathbb{R}^n$$



In case you didn't know, we define velocity as the derivative of the position with respect to time. The reason we use the derivative is that velocity measures how quickly the position changes over time. The derivative gives us this exact "rate of change."

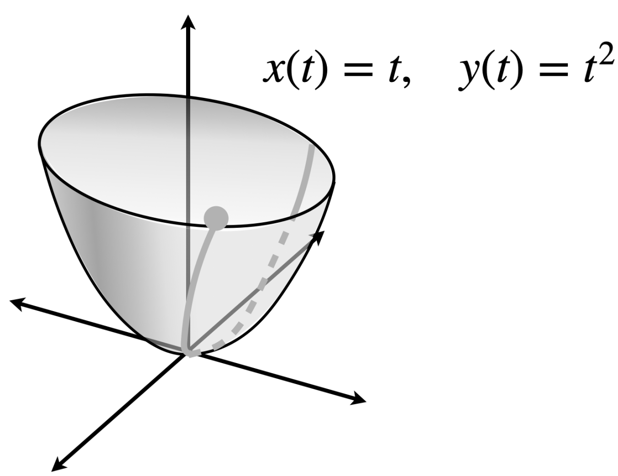
So to find the velocity vector, just like the one we looked at in the beginning, specifically at *p* for example, we take the derivative of $x^p(t)$.

$$\text{"velocity" at } p = \left.\frac{d\varphi \circ \gamma(t)}{dt}\right|_{t=t_0} = \left[\frac{dx^1(t)}{dt}, \ldots, \frac{dx^n(t)}{dt}\right]\Bigg|_{t=t_0}$$

Now let’s see an example.

Consider a paraboloid defined by the equation $z = x^2 + y^2$.

Let’s say the point moves along the following trajectory: $x(t) = t, \;\; y(t) = t^2$.



So the x-coordinate increases linearly with time, and the y-coordinate increases quadratically. The z-coordinate, which depends on x and y, is determined by the paraboloid equation $z(t) = x(t)^2 + y(t)^2 = t^2 + (t^2)^2 = t^2 + t^4$.

Thus, the parametric curve on the surface is given by
$\vec{v}(t) = (\, x(t), y(t), z(t)\,) = (\, t, t^2, t^2 + t^4\,)$

The tangent vector at any point on the curve is given by the derivative of the position vector $\vec{v}(t)$ with respect to *t.*

$\vec{v}(t) = \frac{d\vec{v}(t)}{dt}$ this vector lives in the tangent space.

This means we differentiate each component of the position vector

$$\vec{r}(t) = \left(t, t^2, t^2 + t^4\right)$$

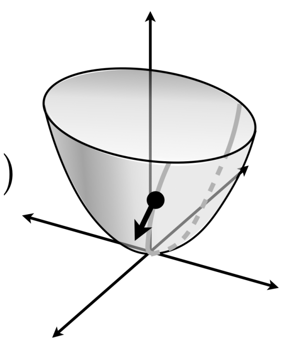
$$1. \frac{dx(t)}{dt} = \frac{d}{dt}(t) = 1$$

$$2. \frac{dy(t)}{dt} = \frac{d}{dt}(t^2) = 2t$$

$$3. \frac{dz(t)}{dt} = \frac{d}{dt}(t^2 + t^4) = 2t + 4t^3$$

So, the tangent vector (or velocity vector) is this. This vector tells us the direction and speed at which the point is moving along the curve at any time t.

$$\vec{v}(t) = \left(1, 2t, 2t + 4t^3\right)$$



The speed of the point along the curve is the magnitude (length) of the tangent vector. To find the magnitude, we use the formula for the Euclidean length of a vector.

$$|\vec{v}(t)| = \sqrt{\left(\frac{dx(t)}{dt}\right)^2 + \left(\frac{dy(t)}{dt}\right)^2 + \left(\frac{dz(t)}{dt}\right)^2}$$

Substituting the components of $\vec{v}(t)$

$$|\vec{v}(t)| = \sqrt{1^2 + (2t)^2 + (2t + 4t^3)^2}$$

Let’s simplify this step-by-step:

$$|\vec{v}(t)| = \sqrt{1^2 + (2t)^2 + (2t + 4t^3)^2}$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + (2t + 4t^3)^2}$$

$$|\vec{v}(t)| = \sqrt{1 + 4t^2 + (2t + 4t^3)^2}$$

$$(2t + 4t^3)^2 = 4t^2 + 16t^4 + 16t^6$$

$$|\vec{v}(t)| = \sqrt{1 + 8t^2 + 16t^4 + 16t^6}$$

$$\vec{v}(t) = \left(1, 2t, 2t + 4t^3\right)$$