# Algebraic Geometry

*By DiBeos*

We go back to the 1600's, to **Desargues' Theorem**, part of projective geometry, where the roots of algebraic geometry lie.

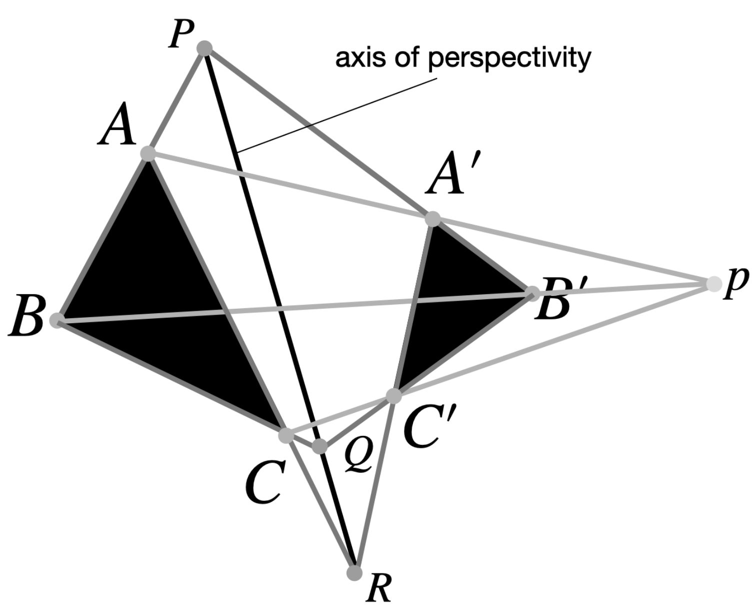
To understand what it is, begin by drawing two triangles. Label the vertices of the first triangle A, B and C, and the vertices of the second triangle A′, B′ and C'.

Draw lines connecting the corresponding vertices of the two triangles. So, from A to A'. From B to B', and from C to C'.

If Desargues' Theorem holds (which it will if the triangles are in a special position called "perspective from a point"), all the lines should meet at a single point, known as the *center of perspectivity*, which we label p.

Now, extend the sides of each triangle to meet the corresponding extended sides of the other triangle. So, extend AB to meet A′B′. BC to meet B′C′. CA to meet C′A′, and label each intersection point (P as the intersection of AB and A′B′. Q as the intersection of BC and B'C'. R as the intersection of CA and C'A').

According to this Theorem, the points P, Q, and R will lie on a straight line, known as the *axis of perspectivity*.



About 30 years later came **Pascal's Hexagrammum Mysticum Theorem**, which showed deeper implications in the study of properties of conic sections and the relationships between points and lines.

To understand how, draw a conic section, which can be a circle, the bottom of the cone, for simplicity.

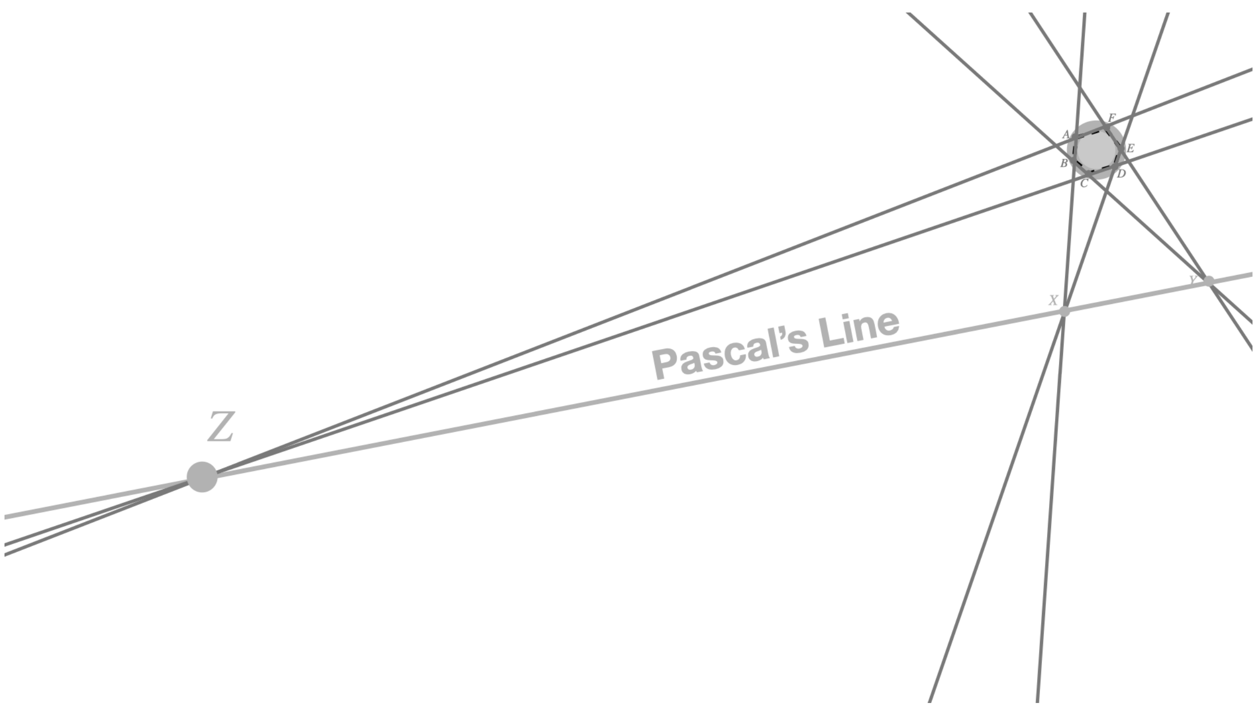
Pick 6 random points on the boundary of the circle. As long as they are distinct, they will form a hexagon, which is not necessarily regular. Label them A, B, C, D, E and F. Let's keep the order of A being the first, and F the last.

Now, we draw a line through the points A and B. We likewise do that with D and E. We see that the two lines intersect, and we label the intersection point X.

We do the same thing with points B C and E F. They also intersect, and we label the intersection point Y.

Lastly, we pass a line through C D and F A. Their intersection is a long way away, but does occur, and we label it Z.

According to Pascal, the X Y and Z will have a straight line going through them. This is known as Pascal's line.




But here's a fun fact: there are actually 60 different ways of connecting the lines, resulting in 60 different pascal's lines. The point is, no matter how you connect them, Pascal's line will pass somewhere

This leads us to **Elimination Theory**.

Let's say we have the line $y = mx + b'$, and the parabola $y = ax^2 + bx + c$ .

We seek to determine their points of intersection by eliminating y and solving for x. Hence, replace the value of y in the parabola equation with that given by the line $mx + b' = ax^2 + bx + c$. Collect all terms on one side of the equation to create a single polynomial equation $ax^2 + (b - m)x + (c - b') = 0$.

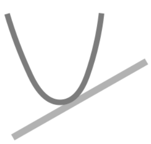
Here, $ax^2 + px + q = 0$ is a general quadratic equation, where $p = b - m$ (the combined coefficient of x), and $q = c - b'$ the constant term after rearranging.

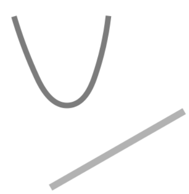
Usually we'd solve it with a quadratic formula, formalized by Descarte in the mid 17th century.

$$x = \frac{-(b-m) \pm \sqrt{(b-m)^2 - 4a(c-b)}}{2a}$$

But, how would mathematicians have solved it *before* the quadratic formula?

→ If $p^2 - 4aq$ is + , the roots are real, intersection 

→ If $p^2 - 4aq$ is $0$ the 

→ If $p^2 - 4aq$ is $-$ , no intersection 

Scholars of the time might have attempted to factorize the quadratic equation manually if possible. They might also have used numerical approximations to estimate the roots.

For example, imagine $y = x + 1$ (a line) and $y = x^2 + 2x + 3$ (a parabola):

Equate the line to the parabola:

$x + 1 = x^2 + 2x + 3$

Then, rearrange terms:

$x^2 + x + 2 = 0$

We would usually use the discriminant formula here, $\Delta = b^2 - 4ac$. But, assuming they didn't have that yet, mathematicians would have attempted to solve it by *completing the square* for example.

$$0 = x^2 + x + 2$$

$$-2 = x^2 + x$$

$$\left(\frac{1}{2}\right)^2 + (-2) = x^2 + x + \left(\frac{1}{2}\right)^2$$

$$-\frac{7}{4} = \left(x + \frac{1}{2}\right)^2$$

The left-hand side represents the square of a number, which cannot be negative. Therefore, the equation has no real solutions, and it means that the line does not intersect the parabola.

This method is indeed much more powerful than the one before it for computing things. But it's still computationally intensive and complex. Bringing forth **Abstract Algebra.**

The first and simplest type is this: $ax^2 + bx + c = 0$. Solving the equation means finding the values of x that make the equation true. These values are called roots.

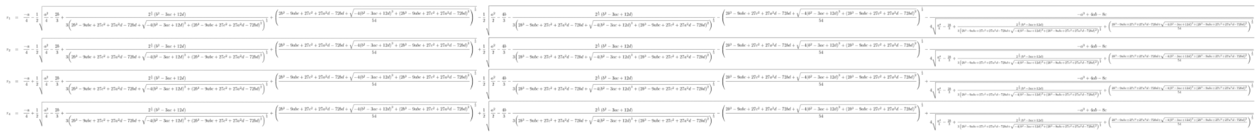
To find these roots, 19th century mathematicians found this quadratic formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Then, when it's a cubic equation, $at^3 + bt^2 + ct + d$, mathematicians found this long and complicated formula.

$$\frac{\sqrt[3]{-27a^2d+9abc-2b^3+3a\sqrt{3(27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2)}} + \sqrt[3]{-27a^2d+9abc-2b^3-3a\sqrt{3(27a^2d^2-18abcd+4ac^3+4b^3d-b^2c^2)}}}{3\sqrt[3]{2}a}.$$

By the time they got to quartics, this even more complicated formula was found



By the time of Abel and Galois, solutions for quadratic, cubic, and quartic equations were well-established using radicals (or expressions involving roots). However, Abel, and later Galois, proved that the general quintic (the degree 5 polynomial) cannot be solved this way.

There is no such quadratic formula for quintic roots

$$x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0 = 0$$

Although we did make a brief explanation in this video here

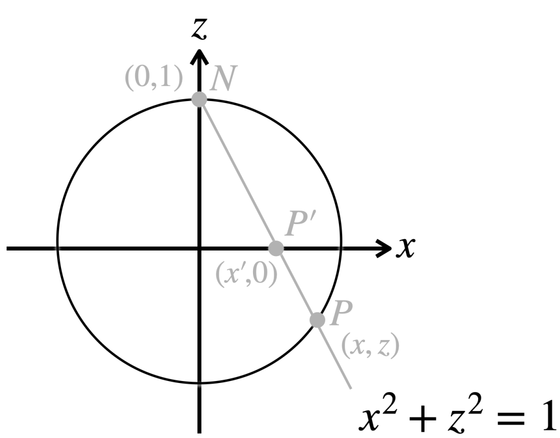
What is Solvability in Galois Theory?

Which is where **Birational Geometry** comes in.

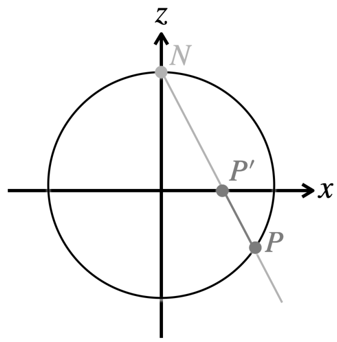
One concept within it is stereographic projection, which is a way to "map" points on a circle (or a sphere, in higher dimensions) to points on a line (or a plane). This mapping helps us relate the geometry of a curved object (like a circle) to a simpler, flat object (like a line).

Consider the unit circle in the plane, given by the equation $x^2 + z^2 = 1$. The circle lies in the *xz*-plane, centered at the origin (0, 0), with a radius of 1.

In this diagram, there is a “North Pole”, which is the projection point from which we will construct a *birational map.* Pick any point P with coordinates (x, z) on the circle. And draw a straight line from N through P. Extend this line until it intersects the x-axis.

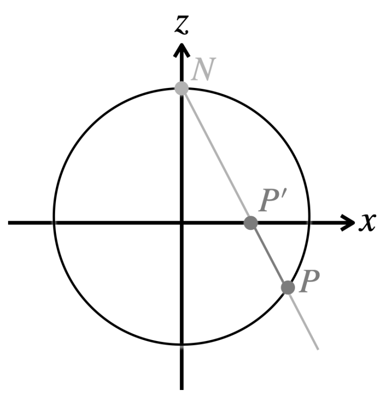


The coordinates of P' can be determined by solving the equation of the line for where it intersects z = 0. So, P on the circle maps to P' on the line using this formula.



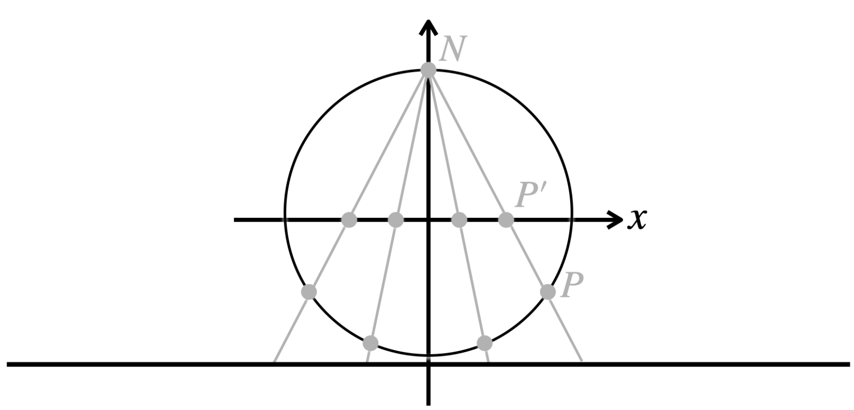

$$x' = \frac{x}{1 - z}$$

From a point P' with coordinates (x', 0), we can reverse the projection to find the corresponding point P with coordinates (x, z) on the circle. These are the inverse formulas




$$x = \frac{2x'}{1 + x'^2}, \quad z = \frac{1 - x'^2}{1 + x'^2}$$

The stereographic projection shows that every point on the circle (except for the north pole) can be mapped to a unique point on the line, and vice versa. The mapping and its inverse are given by rational functions (fractions of polynomials), which makes the circle and the line "birationally equivalent" in algebraic geometry.

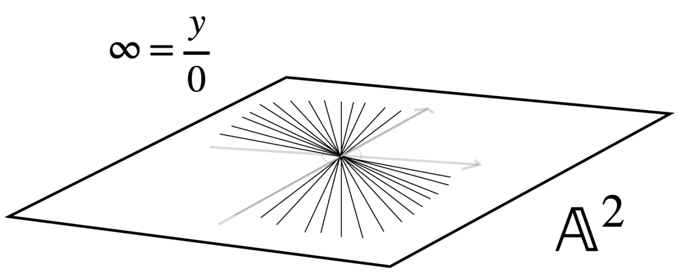


Birational geometry lacked the tools to analyze complex structures and local properties of surfaces in higher dimensions, particularly in terms of their topology and complex analytic properties, which is what **Riemann surfaces and topology** address.

Blowing up and blowing down are operations in algebraic geometry that are particularly significant in the study of surfaces.

We have a space, $\mathbb{A}^2$, which is a 2D coordinate plane with coordinates x and y. We focus on the origin (0,0).

At the origin is a point. From this point we can draw infinite lines that pass through this point, each defined by a slope *m = y/x*. However, the slope *m = y/x* becomes problematic for vertical lines because when x = 0, it results in m = $\infty$




Thus, to handle this, we have to blow up the origin. By blowing up the origin, we replace the single point (0, 0) with points unique to each individual line and thus create a projective line $\mathbb{P}^1$. By doing so, we introduce projective coordinates *[u : v]*, where the slope *m = u/v* captures the slope of each line.