Abstract Algebra Origins

We open by introducing natural numbers, denoted as

$$\mathbb{N} := \{1,2,3,\ldots\}$$

These numbers form the basis of arithmetic operations such as addition and multiplication, fundamental to all of mathematics. As the narrative unfolds, we delve into Diophantine equations, first proposed by Diophantus of Alexandria. A typical example of a linear Diophantine equation presented is

$$ax + by = c$$

demonstrating the challenges faced using primitive numerical tools which excluded negative numbers and zero.

A classic method known to mathematicians of later periods, but possible in an early form for Diophantus, is to express the sides in terms of two parameters m and n

$$\begin{aligned} a &= m^2 - n^2 \\ b &= 2mn \\ c &= m^2 + n^2 \end{aligned}$$

As mathematical thought evolved, the introduction of zero and negative numbers, symbolized as integers, marked a significant advancement.

$$\mathbb{Z} := \{-2, -1,0,1,2,\ldots\}$$

The narrative then shifts to the contributions of Al-Khwarizmi, who is credited with laying the groundwork for modern algebra through his pioneering techniques for solving equations of the form

$$ax^2 = bx$$

and

$$ax^2 + c = bx$$

The necessity for rational numbers is highlighted next,

$$\mathbb{Q} := \left\{ \frac{a}{b} \mid a, b \in \mathbb{Z}, b \neq 0 \right\}$$

illustrating their importance through equations that cannot be solved using integers alone, such as

$$2x = 1$$

This leads into the discussion of symbolic algebra introduced by François Viète, who utilized letters to represent both known and unknown quantities.

We further explore the intersection of algebra and geometry through René Descartes' analytical geometry. Descartes introduced the Cartesian coordinate system, enabling polynomial equations to be represented as geometric curves.

This segment also covers Descartes' Rule of Signs, a method to estimate the number of positive and negative real roots of a polynomial based on the sign changes between consecutive terms of the polynomial, expressed as

$$P(x) = x^3 - 6x^2 + 11x - 6$$

Descartes' Rule of Signs says that we need to count the sign changes between consecutive terms.

****It's important to note** that in the video we made the mistake of saying there are two signs, but there are actually three, the third being between *-6* and *+11*.

Isaac Newton's contribution through the Newton-Raphson method is discussed, providing a numerical approach to finding roots of polynomials by iteratively refining guesses, demonstrated through the formula

$$x_{n+1} = x_n - \frac{P(x_n)}{P'(x_n)}$$

The first step in applying Newton's method is to compute the derivative of the polynomial. Which is this:

$$P'(x) = 3x^2 - 12x + 11$$

In this example, we start with an initial guess of

$$x_0 = 2.5$$

Now, we need to calculate the value of the polynomial and its derivative

$$P(2.5) = 2.5^3 - 6 \cdot 2.5^2 + 11 \cdot 2.5 - 6 = -0.375$$

$$P'(2.5) = 3 \cdot 2.5^2 - 12 \cdot 2.5 + 11 = 2.75$$

Update x using the Newton-Raphson formula, and we get this

$$x_1 = 2.5 - \frac{-0.375}{2.75} = 2.5 + \frac{0.375}{2.75} \approx 2.6364$$

To get closer to the exact root, you would repeat the process using the result.

We then transition to Galois Theory, which examines the solvability of polynomial equations: through symmetrical operations and introduces group theory. Essential properties of a group such as closure, associativity, identity, and inverses are defined.

Closure: For any two elements a and b in the group, the result of the operation $a \cdot b$ is also in the group.

Associativity: For all *a*, *b* and *c* in the group, the operation satisfies:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Identity Element: There exists an element *e* in the group such that for every element *a*, the operation satisfies

$$e \cdot a = a \cdot e = a$$

Inverse Elements: For every element a in the group, there exists an inverse element *b* such that the operation satisfies

$$a \cdot b = b \cdot a = e$$

Finally, we conclude with a discussion on rings and fields, detailing their structural properties. Rings combine two operations—addition and multiplication—with rules like distributivity, while fields extend these concepts by requiring commutativity and the existence of multiplicative inverses for non-zero elements.

Within a ring, the addition operation must form an Abelian group. This means that the addition operation is:

Associative: For all *a*, *b*, and *c* in the set,

$$a + (b + c) = (a + b) + c$$

Commutative: For all *a* and *b* in the set

$$a + b = b + a$$

Has an identity element (zero): There exists an element *0* such that for every element *a*

$$a + 0 = 0 + a = a$$

Each element has an additive inverse: For every element *a*, there exists an element *-a* such that

$$a + (-a) = 0$$

In the context of a ring, the multiplication operation is associative, meaning that for all *a*, *b*, and *c* in the set, the grouping of multiplications does not change the outcome:

$$(a \times b) \times c = a \times (b \times c)$$

Multiplication must distribute over addition both from the left and the right, ensuring that operations like factoring and expansion work similarly to how they do with regular numbers.

Left Distributivity: For all *a*, *b*, and *c* in the set,

$$a \times (b + c) = a \times b + a \times c$$

Right Distributivity: For all *a*, *b*, and *c* in the set,

$$(a + b) \times c = a \times c + b \times c$$

And this is how we get to Abstract algebra. Consider the polynomial equation:

$$P(x) = x^5 - x^3 + 2x^2 - x + 3 = 0$$

In the field of real numbers, denoted as $\mathbb{R}$, or even in the field of rational numbers $\mathbb{Q}$, solving such high-degree polynomial equations exactly is generally impossible using elementary algebraic methods or operations defined within the structures of rings and fields alone.