7 Indeterminate Forms

When the denominator of a fraction becomes very small—approaching but not reaching zero—the result of the division becomes very large.

$$\frac{1}{0.5} = 2 \qquad \frac{1}{0.1} = 10$$

$$\frac{1}{0.01} = 100 \qquad \frac{1}{0.001} = 1000$$

This pattern suggests that as the denominator gets infinitesimally close to zero, the fraction itself appears to grow toward infinity.

But in whatever context it is used, division by zero is meaningless.

Suppose we try to divide *5* by *0*. Say we call the hypothetical outcome *x*.

In order for that to be true, so must this expression $0 \cdot x = 5$. But, anything times *0* is equal to zero, so the outcome, whatever value we pick for *x*, would be *0=5*. Which is of course, not true.

We go on to discuss why multiplication by *0* results in *0*.

Multiplication is fundamentally repeated addition. Each term you're adding is the multiplicand, and you're adding it as many times as the multiplier indicates.

When you multiply $0 \cdot 5$, you're essentially adding *5* zero times. Since no addition occurs, the result is 0. On the other hand, $5 \cdot 0$ means adding *0* together *5* times. Even though addition occurs, since each term added is *0*, the result remains *0*.

That leaves us at *0=5* being an absurd result, and therefore, we also cannot have *5* divide *0*. But what about dividing *0* by *0*?
Again, this puts us at $0 \cdot x = 0$.

x can be absolutely any number. A unique answer does not exist in this case. So, the operation is declared as invalid.

We go back full circle to the expression $\frac{1}{0} = \infty$, since 1 divided by 0 is meaningless.

We highlight why *1* is also important for the relationship between *0* and ∞.

This number generates all positive integers by successive addition. Thus all three elements have three unique roles on the number line – the *0* is the starting point, *1* is the scale used, and ∞ shows us the completeness of the line, since it includes all real numbers.

Together, these three lead us to a very interesting concept in mathematics, known as indeterminate forms.

We've already discussed one of them → $\frac{0}{0}$. That's an indeterminate form.

$\frac{0}{0}$ is the same thing as saying:

$$\frac{\frac{1}{\infty}}{\frac{1}{\infty}}$$

Which can be transformed into:

$$\frac{1}{\infty} \qquad \frac{\infty}{1}$$

Which simplifies to:

$$\frac{\infty}{\infty}$$

Thus, infinity over infinity is an equivalent of *0/0* and is an indeterminate form.

$$\frac{\infty}{\infty} = \frac{0}{0}$$

In order to explore its limits, let's say we have the expression:

$$\frac{2x+1}{x-1}$$

There are infinite values we can pick, so let's pick all of the possible ones and say that *x* tends to infinity.
Our intuition might tell us that we should substitute $x\ for\ \infty$. When we do, everything just results in infinity over infinity.

We need to simplify the expression by dividing x from the top and bottom

$$\lim_{x\to\infty} \frac{2+\frac{1}{x}}{1-\frac{1}{x}}$$

Remember, since *x* tends towards infinity the *x* becomes infinity. So $\frac{1}{\infty}$ is the same as saying *0*.

$$\lim_{x\to\infty} \frac{1}{x} = 0$$

Since *2+0* is *2* and *1-0* is *1*, the answer is *2*.

$$\lim_{x\to\infty} \frac{2+0}{1-0} = \lim_{x\to\infty} \frac{2}{1} = 2$$

Despite x tending towards infinity, we found that it will not surpass the 2 limit.

Another Indeterminate form is $\infty - \infty$.

It might be tempting to say that anything minus itself results in zero. But $\infty - \infty$ doesn't mean that it is "the number minus the same number" because ∞ doesn't represent any fixed number. So we cannot say that $\infty - \infty = 0$, rather, its value cannot be determined, therefore it is indeterminate.

The problem arises because it's unclear how fast each "infinity" is reached and whether these rates are comparable.

Consider the limit of the difference between two functions as x approaches infinity

$$\lim_{x\to\infty} (x^2 - x)$$

As $x \to \infty$, x^2 increases much faster than *x* .Thus, x^2 and *x* both tend towards infinity, but at different rates. The x^2 grows quadratically, so it dominates the linearly increasing *x*.

We can still simplify the expression $x^2 - x = x(x - 1)$
As *x* becomes very large, the term $x - 1$ is almost the same as x (since subtracting *1* becomes negligible at high values). Hence, the product *x(x - 1)* is approximately x^2, which clearly tends to infinity.

The next indeterminate form is $0 \cdot \infty$.

We will again use the same $\frac{0}{0}$ form to prove this. We already know that *1/0* = ∞. Now, *0/0 = 0 × 1/0 = 0 × ∞* . Since $\frac{0}{0}$ is an indeterminate form, *0 × ∞* is also an indeterminate form.

Consider this limit:

$$\lim_{x \to \infty} \left(\frac{1}{x} \cdot x^2 \right)$$

Here, despite the zero factor and the infinity factor, the result is an infinite limit. This example shows that the zero factor doesn't necessarily "win" over the infinite growth of the second factor, depending on how these rates of change balance out.

The next indeterminate form is ∞^0

Intuition tells us it's quite obvious – anything raised to the power of *0* is *1*. The operation ∞^0 means taking something that grows without bounds and raising it to a power that traditionally nullifies growth (turning everything into one).

The core issue here is defining how a boundless base is restrained by a zero exponent, which inherently lacks clarity without a limiting process.

We know that a^0 using the quotient rule of exponents can be written as a/a. In the same way, ∞^0 can be written as ∞/∞, which is an indeterminate form. Thus, ∞^0 is an indeterminate form.

As always, infinity to the power of zero will lead to a different answer based on the conditions we set up.

The next indeterminate is 1^∞

At a superficial level, one might think *1* raised to the power of *1* should remain *1*, since it's *1* multiplied by itself an infinite number of times. But, since infinity is not a number, we have to treat it like a function that infinitely approaches *1*.

If we take a number that is less than and very close to *1*, then multiplying it by itself an infinite number of times, gives a very very small number and is approximately equal to *0*.

If we take a number that is greater than and very close to *1*, then multiplying it by itself an infinite number of times, gives a very very bigger number and is approximately equal to ∞.

It means, the limit *lim x → 1*, x∞ does not exist because its left-hand limit is 0 and the right-hand limit is ∞. Hence, if we get 1∞ after the substitution into the limit, it means that we have got an indeterminate form.

left-hand limit **right-hand limit**

$$\lim_{n\to\infty} 0.999^n = 0 \qquad \lim_{n\to\infty} 1.001^n = \infty$$

$$x = 0.999 \qquad x = 1.001$$

The last limit is 0^0

Since any number a to the power of *0* is equal to 1

$$a^0 = 1$$

This should therefore mean that 0^0 is *1*. But also, if we say that *0* to the power of a will always be *0*, it would mean that *0* to the power of zero equals to *1* is the same as saying as *0* to the power of *0* is equal to *0*, which of course it isn't.

But, when both the base and the exponent are zero, these rules conflict. The question becomes whether the "all powers of zero result in zero" rule overrides the "anything raised to the zero power equals one" rule, or vice versa.

We need more in order to determine the value. Take this limit:

$$\lim_{x\to 0^+} x^x$$

It evaluates to:

$$x^x = e^{x\ln(x)}$$

As $x \to 0^+$, $x\ln(x) \to 0$ (because $\ln(x)$ approaches $-\infty$ slower than *x* approaches *0*), leading to:

$$\lim_{x\to 0^+} e^{x\ln(x)} = e^0 = 1$$