Simple Finite Groups

We start out by discussing the classification of finite simple groups, a major achievement in modern mathematics. Although much progress was made by the 1980s, the final gaps in the classification were only filled in 2004 with the introduction of quasithin groups.

To make these abstract ideas more approachable, the discussion turns to a simple algebraic equation, $x + 5 = 7$, and how its solution illustrates key concepts in group theory.

$x + 5 = 7$
$(x + 5) + (-5) = 7 + (-5)$
$x + (5 + (-5)) = 2$
$x + 0 = 2$
$x = 2$

We see that the integer 0 is the identity element. This means adding 0 to any number leaves the number unchanged $x + 0 = x$. The existence of this identity element is crucial for solving the equation and isolating x.

To subtract 5, we actually add its inverse: -5. For each element in a group, there must be an inverse such that their sum equals the identity. Here, $5 + (-5) = 0$, illustrating the concept of inverse elements.

The order in which the numbers are grouped is irrelevant. This reflects the property of associativity, which means that how numbers are grouped during addition doesn't affect the result $x + (5 + (-5)) = (x + 5) + (-5)$.

The set of integers is closed under addition. This means that adding any two integers will always result in another integer $7 + (-5) = 2$.

However, when we consider multiplication, things change. Not all real numbers form a group under multiplication because 0 does not have a multiplicative inverse (since no number multiplied by 0 results in 1). The identity element is 1 for multiplication (since multiplying by 1 leaves the number unchanged).

But what happens when we extend our equation into something more complex?
"Two lefts and what is a right" denoted as \(2L + x = R\).

This appears similar to algebra, with numbers and an operation (addition), but now it involves directions.

Left (L), Right (R), Back (B), and Forward (F). These actions correspond to specific rotations of 90, 270, 180, and 0 degrees, respectively.

2L represents a total rotation of *180* degrees, which is equivalent to a back turn (B). The equation becomes *B* + *x* = *R*.
Since *180* + *90* = *270*, the solution is that *x* = *L* (a left turn). This shows how group operations can represent rotations and how they adhere to the properties of groups.

We then introduce Cayley tables. Each row and column in the Cayley table represents a group element. The cell at the intersection of a row and a column shows the result of applying the group operation to those two elements.

*	F	L	B	R
F	F	L	B	R
L	L	B	R	F
B	B	R	F	L
R	R	F	L	B

Each element like *L* (*Left 90 degrees*) or *R* (*Right 270 degrees*) interacts with others, and the table shows the outcome. If *L* * *R*=*F*, it means *L* and *R* are inverses since their operation results in the identity element *F*.

By comparing two Cayley tables side by side, we see that if the elements *F, L, B* and *R* in Table 1 are replaced with *0, 1, 2,* and *3* respectively, the groups represented by these tables are isomorphic.

*	F	L	B	R
F	F	L	B	R
L	L	B	R	F
B	B	R	F	L
R	R	F	L	B

$+_4$	0	1	2	3
0	0	1	2	3
1	1	2	3	0
2	2	3	0	1
3	3	0	1	2

An isomorphism occurs when you can remap the elements of one group to another, while preserving the operation outcomes.

An isomorphism is a bijection, meaning it is both:
Injective (One-to-One): Every element in the first group maps to a unique element in the second group without repetitions.
Surjective (Onto): Every element in the second group corresponds to an element in the first group.

We introduce homomorphisms, which are a broader and more flexible idea than isomorphisms. A homomorphism can be injective, surjective, or neither.

If you have two groups *G* and *H*, a function $f : G \to H$ is a homomorphism if, for every pair of elements *a* and *b* in *G*, the following condition holds:

$$f(a * b) = f(a) \circ f(b)$$

Just like integers can be broken down into factors, groups can also be broken down or understood through a similar process. One way to do it is through subgroups.

Subgroups allow us to study the internal divisions within a group. By identifying subgroups, we gain insight into how the larger group is structured.

While subgroups provide insight into internal divisions, cosets help us explore how these divisions influence the arrangement of the entire group.

A coset is a subset formed by taking a subgroup *H* and multiplying all of its elements by a particular element *g* from the larger group *G*. Depending on whether you multiply *g* on the left or the right, you get:
Left Cosets: *gH*, where each element of *H* is multiplied on the left by *g*.
Right Cosets: *Hg*, where each element of *H* is multiplied on the right by *g*.

We go on to Lagrange's Theorem, which states that the order (or number of elements) of any subgroup *H* of a finite group *G* divides the order of *G*. Mathematically, this is expressed as:

$$|G| = |H| \times [G : H]$$

For example, if group *G* has 12 elements and subgroup *H* has *4* elements, then according to Lagrange's Theorem, there are *12/4 = 3* cosets of *H* in *G*, with each coset containing *4* elements, thus partitioning the group perfectly.

A particularly interesting type of subgroup is the normal subgroup. What makes normal subgroups unique is that their left cosets and right cosets are identical.

We present an example using the group of integers under addition, denoted $\mathbb{Z}$ and the subgroup $4\mathbb{Z}$, which consists of all multiples of *4* (such as ..., *-8, -4, 0, 4, 8*, ...). For this subgroup, we can form cosets by adding a fixed number to each element of the subgroup.

1. $0 + 4\mathbb{Z} = \{..., -8, -4, 0, 4, 8, ...\}$
2. $1 + 4\mathbb{Z} = \{..., -7, -3, 1, 5, 9, ...\}$
3. $2 + 4\mathbb{Z} = \{..., -6, -2, 2, 6, 10, ...\}$
4. $3 + 4\mathbb{Z} = \{..., -5, -1, 3, 7, 11, ...\}$

To determine if $4\mathbb{Z}$ is a normal subgroup, we need to check if the left cosets and right cosets are the same.

Left coset: $n + 4\mathbb{Z} = \{n + 4k \mid k \in \mathbb{Z}\}$

Right coset: $4\mathbb{Z} + n = \{4k + n \mid k \in \mathbb{Z}\}$

Since these cosets are the same, it is a normal subgroup.

Normal subgroups allow groups to be broken down into simpler components called quotient groups. A quotient group is formed by taking all the cosets of a normal subgroup like $4\mathbb{Z}$ and treating each coset as a single element in a new group. The set of all these cosets forms the quotient group $\mathbb{Z}/4\mathbb{Z}$.

We introduce the Extension Problem in group theory, which arises from this curiosity about construction. The problem asks: given a simpler subgroup and a quotient group, in how many ways can we combine these to form other, potentially more complex groups?

This is a significant and unsolved problem in modern mathematics.

The Classification of Finite Simple Groups was completed in 2004. It resulted in a complete list of all possible finite simple groups, categorized into several families based on their structures and properties.