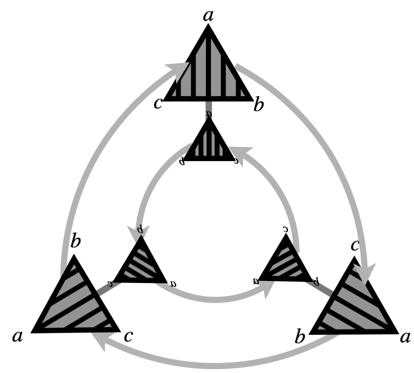
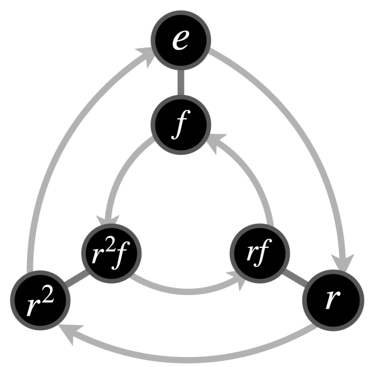
Lagrange's Theorem

We can create a group by taking an object and performing various rotations and flips on it, resulting in the group C_3 when we just rotate it, and S_3 when we include flips.



Each triangle can be replaced with nodes and corresponding letters, *r* for rotation, *f* for flip, and *e* for what is known as the identity element, the original position or the starting point.

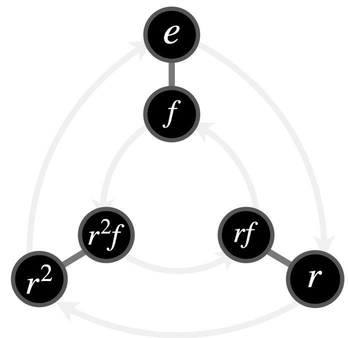


But, the initial group we observed, C_3, is still clearly inside, and acts exactly the same as C_3 would by itself. We don't ignore it, and instead label it as the *subgroup* of S_3, written like $C_3 < S_3$.

C_3 is composed of $\{e,\ r,\ and\ r^2\}$, but it's technically *only* the action of rotation. So instead, we can write it as $C_3 = < r >$.

There is another subgroup within S_3 – C_2, which is a flip between the elements $\{e,\ f\}$. Since it is just the action of flipping, we label it as $< f >$.

What about the other copies of *f* which appear throughout the rest of the diagram? ***(Consider becoming a member of the channel!) Thanks!***



A subgroup, because it is a group, must have an identity element. So $\{e,\ f\}$ is the subgroup C_2. The others are cosets.

If you observe closely, any type of group, every subgroup will have a coset, and these will cover every node of the diagram.

So far, we've mostly looked at cosets visually, but we can describe them algebraically just as well. Say we want to generate the entire coset $< f >$ and start at the node....r, for example. How would we do it?

We multiply the element r by the list of elements of $< f >$. r times e is r by the way, since when you multiply an element by the identity it results in the element.

$$r\langle f\rangle = r\{e,f\} = \{r \cdot e\,,\ r \cdot f\} = \{r,\ rf\}$$

And that's how visually, we end up generating this coset $r < f >$.

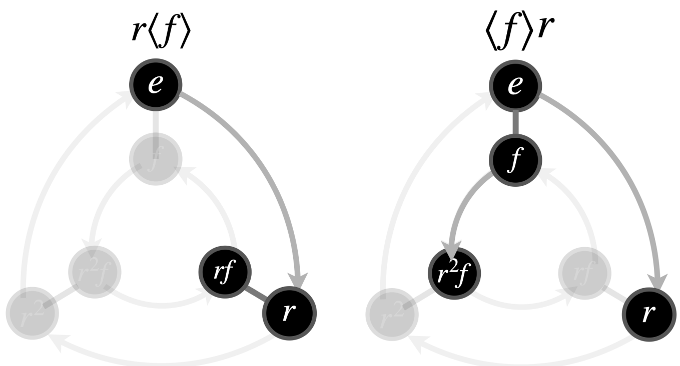
Each coset can actually have more than one name. In the initial case, we can say that it is the copy of $< f >$ based at r . But it is equally true if we say that it is the copy of $< f >$ based at rf.

The fact that we put r on the left side of $< f >$ is also not a coincidence, by the way – it's actually more appropriately called a left coset. A right coset then, is $< f > r$. And, instead of multiplying from the left, as we have done previously, the multiplication is done on the right.

$$\langle f\rangle r = \{e,f\}r = \{e \cdot r\,, f \cdot r\} = \{r,\ r^2 f\}$$

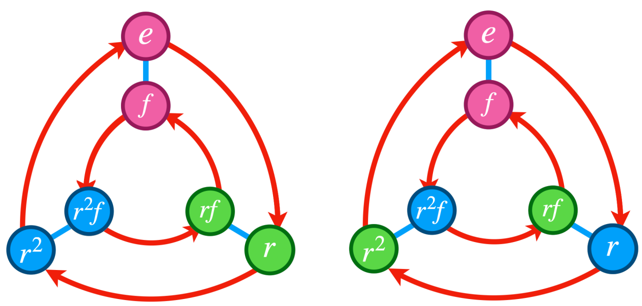
$$r\langle f\rangle = r\{e,f\} = \{r \cdot e\,,\ r \cdot f\} = \{r,\ rf\}$$

If you compare it to the previous calculation, you'll see that they actually don't match at all. They also look different on a Cayley Diagram (end up at different nodes).



Every element in the group will appear in exactly one coset.

In this diagram we see that not only do the left cosets of $< f >$ or $\{e, f\}$ cover all of S_3, but also that no cosets nor their elements overlap. This is equally true for the right cosets of $< f >$.



Is this coincidence? Do the cosets of the subgroup always partition the subgroup so perfectly? The answer to this lies in Lagrange's Theorem.

If we have the subgroup H in a group G, then each element of G will belong to exactly *one* left coset of H. Suppose we have an element g in the group G. And for some reason it looks to us like it belongs to both the left coset aH and bH.

If g is in aH, it is therefore appropriate to say that $gH = aH$, as we observed earlier. And since it is likewise in bH, then $gH = bH$. In that case, $aH = bH$, and both are equal to gH. This means that aH and bH are actually just two different names for the same coset.

We've therefore established that G can be perfectly partitioned into copies of the subgroup H.

When we say $|G|$ to represent groups, we imply their size to be finite. How big *G* is can be established just by counting how many copies of *H* there are, which is *n*. And so *G* is the same as multiplying that *n* by *H*.

$$|G| = \overbrace{|H| + |H| + \ldots + |H|}^{n \text{ times}}$$

n has a formal name: an index.

To formalize what we've established, if *H* is the subgroup of *G* then the index of *H* in *G* we we instead write as $[H: G]$, means how many times $|H|$ goes into $|G|$. So the index is *n*, which is the total number of left cosets of *H*, if we count *H* itself as a coset.

Consider the example $< f > \; < S_3$.

The order, or number of elements, of S_3 is 6, which is composed of 3 cosets (including the subgroup) of order 2 (meaning that all three cosets contain 2 elements each). Thus the index (or the *n*), we established earlier is 3.

$$[S_3 : \langle f \rangle] = \frac{S_3}{\langle f \rangle} = \frac{6}{2} = 3$$

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Based on the book Visual Group Theory by Nathan Carter.